Physics

Composition of two Simple Harmonic Motion

Composition of two SHM of same frequency but different phase and amplitude:

Let a particle in a medium be simultaneously acted upon by two Simple Harmonic vibrations of same frequency but different phase and amplitude given by the following equation

$$\mathbf{y_1 = a_1 \sin(\omega t + \alpha_1)} \qquad \textbf{(1)}$$

$$\mathbf{y_2 = a_2 \sin(\omega t + \alpha_2)} \qquad \textbf{(2)}$$

The resultant displacement y of the particle will be

$$\mathbf{y = y_1 + y_2}$$

$$\mathbf{= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2)}$$

$$\mathbf{= a_1(\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + a_2 \sin(\omega t + \alpha_2)}$$

$$\mathbf{= a_1(\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + a_2 (\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2)}$$

$$\mathbf{= (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin \omega t + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos \omega t}$$

Here, a_1, a_2, α_1 & α_2 *are constant*

So, putting $\mathbf{a_1 \cos \alpha_1 + a_2 \cos \alpha_2 = A \cos \phi}$

$$\mathbf{a_1 \sin \alpha_1 + a_2 \sin \alpha_2 = A \sin \phi}$$

The resultant displacement can be written as,

$$\mathbf{y = A \cos \phi \sin \omega t + A \sin \phi \cos \omega t}$$

$$=> y = A\, \mathbf{\sin(\omega t + \phi)} \qquad \dots \dots \dots \textbf{(3)}$$

Which is also simple harmonic motion.

To find the values of A & ϕ, We can write,

$$\mathbf{A^2 \sin^2 \phi + A^2 \cos^2 \phi = a_1^2 \sin^2 \alpha_1 + a_2^2 \sin^2 \alpha_2 + 2a_1a_2 \sin \alpha_1 \sin \alpha_2 + a_1^2 \cos^2 \alpha_1 + a_2^2 \cos^2 \alpha_2 + 2a_1a_2 \cos \alpha_1 \cos \alpha_2}$$

$$=> \mathbf{A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\alpha_1 - \alpha_2)}$$

$$\& \quad A = \sqrt{(}\mathbf{a_1^2 + a_2^2 + 2a_1a_2 \cos(\alpha_1 - \alpha_2))}$$

$$\therefore \mathbf{\tan \phi = \frac{A \sin \phi}{A \cos \phi} = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}}$$

Example 2.1: Two simple harmonic motion acting simultaneously on a particle are given by the equation

$$\boldsymbol{y_1 = 2\sin\left(\omega t + \frac{\pi}{6}\right)} \ \ \& \ \ \boldsymbol{y_2 = 3\sin\left(\omega t + \frac{\pi}{3}\right)}$$

Calculate (i). Amplitude (ii). Phase constant (iii). Time period of the resultant vibration.

What is the equation of the resultant vibration?

Solution:

We know,

$$y_1 = 2\sin\left(\omega t + \frac{\pi}{6}\right) \qquad (1)$$

& $$y_2 = 3\sin\left(\omega t + \frac{\pi}{3}\right) \qquad (2)$$

The equations are similar to the equations

$$y_1 = a_1 \sin(\omega t + \alpha_1)\ \&$$

$$y_2 = a_2 \sin(\omega t + \alpha_2)$$

The equation of the resultant vibration is given by

$$y = A\sin(\omega t + \phi)$$

Where

$$\mathbf{A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\alpha_1 - \alpha_2)}\ \&\ \phi = \tan^{-1}\frac{a_1\,\mathrm{si}\quad{}_1 + a_2\sin\alpha_2}{a_1\cos\alpha_1 + a_2\cos\alpha_2}$$

Here, $a_1 = 2, a_2 = 3, \alpha_1 = \frac{\pi}{6}, \alpha_2 = \frac{\pi}{3}$

(i) $$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\alpha_1 - \alpha_2)}$$

$$=> A = \sqrt{4 + 9 + 2.2.3.\cos(\frac{\pi}{6} - \frac{\pi}{3})}$$

$$\therefore\ \ A = 4.939$$

(ii) $$\phi = \tan^{-1}\left(\frac{2.\sin\frac{\pi}{6} + 3.\sin\frac{\pi}{3}}{2.\cos\frac{\pi}{6} + 3.\cos\frac{\pi}{3}}\right)$$

$$= \tan^{-1}(1.114)\ = 48.1^o = \frac{4\pi}{15}$$

$\therefore$ Phase constant $\omega t + \phi = \omega t + \frac{4\pi}{15}$

(iii) The resultant time period is the same as the time period of the individual vibrations. The equation of the resultant vibration is

$$y = 4.939 \sin\left(\omega t + \frac{4\pi}{15}\right)$$

Derive the expression for two SHM at right angles to each other and having time periods 1:2.

Solution

Consider two SHMs of same frequency but of amplitude a and b and having their vibrations mutually perpendicular to each other. If ϕ is the phase difference between the two vibrations then their equations can be written as

$$\boldsymbol{x = a\sin(2\omega t + \phi)} \qquad \textbf{(1)}$$

$$\boldsymbol{y = b\sin(\omega t)} \qquad \textbf{(2)}$$

Where a and b are the amplitude of the motion along x & y directions respectively and ϕ is phase difference between two vibrations

Equation (1) implies $\boldsymbol{x/a = \sin(2\omega t + \phi)}$

$$= \boldsymbol{\sin 2\omega t cos\phi} + cos\boldsymbol{2\omega t\ sin\phi}$$

$$= \boldsymbol{2\sin\omega t cos\ \omega t cos\phi} + (1 - 2sin^2\boldsymbol{\omega t})\ \boldsymbol{sin\phi}$$

$$= \boldsymbol{2\sin\omega t}\sqrt{(1 - sin^2\boldsymbol{\omega t})}\boldsymbol{cos\phi} + (1 - 2sin^2\boldsymbol{\omega t})\ \boldsymbol{sin\phi}$$

$$= \mathbf{2y/b}\sqrt{(1 - (\mathrm{y/b})^2)}\boldsymbol{cos\phi} + (1 - 2(\mathrm{y/b})^2)\ \boldsymbol{sin\phi}$$

$$\boldsymbol{or}, \frac{\boldsymbol{x}}{\boldsymbol{a}} - (1 - 2(\mathrm{y/b}\,)^2)\ \boldsymbol{sin\phi} = \mathbf{2y/b}\sqrt{(1 - (\mathrm{y/b})^2)}\boldsymbol{cos\phi}$$

or, $\frac{x}{a} - \ \boldsymbol{sin\phi} + 2y^2/b^2\ \boldsymbol{sin\phi} = \mathbf{2y}\boldsymbol{cos\phi}/\mathbf{b}\sqrt{(1 - y^2/b^2)}$

Squaring both sides we have

$$(\boldsymbol{x/a} - \ \boldsymbol{sin\phi})^2 + 4y^4/b^4\ \boldsymbol{sin^2}\phi + 2(\boldsymbol{x/a} - \ \boldsymbol{sin\phi})2y^2/b^2\ \boldsymbol{sin\phi}$$
$$= \mathbf{4}y^2/b^2(1 - y^2/b^2)\boldsymbol{cos}^2\ \phi$$

$$\boldsymbol{or}, (\boldsymbol{x/a} - \ \boldsymbol{sin\phi})^2 + \frac{4y^4}{b^4}(\boldsymbol{sin^2}\phi + \boldsymbol{cos}^2\ \phi) + \frac{4y^2}{b^2}(\boldsymbol{sin^2}\phi + \boldsymbol{cos}^2\ \phi) + \frac{\mathbf{4}y^2}{b^2}.\frac{\boldsymbol{x}}{\boldsymbol{a}}\ \boldsymbol{sin\phi} = 0$$

$$or, (\boldsymbol{x/a} - \ \boldsymbol{sin\phi})^{\boldsymbol{2}} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{\boldsymbol{x}}{\boldsymbol{a}}\ \boldsymbol{sin\phi} - 1\right) = 0$$

which is the general equation of a conic bearing two loops. Actual shape will depend on the phase difference ϕ.

Case I

If $\phi = 0, \pi, 2\pi \quad then\ sin\phi = 0$

Then $\quad \frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0 \quad$ Represents the figure of eight.

Case II

If $\phi = \pi/2 \quad then\ sin\phi = 1$ then

$$(\boldsymbol{x/a} - \ \boldsymbol{sin\phi})^{\boldsymbol{2}} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{\boldsymbol{x}}{\boldsymbol{a}}\ \boldsymbol{sin\phi} - 1\right) = 0$$

Implies

$$(\boldsymbol{x/a} - \ \boldsymbol{1})^{\boldsymbol{2}} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{\boldsymbol{x}}{\boldsymbol{a}} - 1\right) = 0$$

$$or, (\boldsymbol{x/a} - \ \boldsymbol{1})^{\boldsymbol{2}} + \frac{4y^2}{b^2}\left(\frac{\boldsymbol{x}}{\boldsymbol{a}} - 1\right) + \frac{4y^4}{b^4} = 0$$

$$or, \left(\frac{\boldsymbol{x}}{\boldsymbol{a}} - \ \boldsymbol{1} + \frac{2y^2}{b^2}\right)^{\boldsymbol{2}} = 0$$

$$or, \frac{\boldsymbol{x}}{\boldsymbol{a}} - \ \boldsymbol{1} + \frac{2y^2}{b^2} = 0$$

or, $\frac{2y^2}{b^2} = -\left(\frac{\boldsymbol{x}}{\boldsymbol{a}} - \ \boldsymbol{1}\right)$

$or, y^2 = -\frac{b^2}{2}\left(\frac{\boldsymbol{x}}{\boldsymbol{a}} - \ \boldsymbol{1}\right)$

$or, y^2 = -\frac{b^2}{2a}(x - a)$ represents the figure of parabola with vertex (a,0).

2.3 Composition of two SH vibrations at right angles to each other having equal frequencies but differing in phase and amplitude

Consider two SHMs of same frequency but of amplitude a and b and having their vibrations mutually perpendicular to each other. If ϕ is the phase difference between the two vibrations then their equations can be written as

$$\boldsymbol{x} = \boldsymbol{a}\sin(\boldsymbol{\omega t} + \phi) \qquad \textbf{(1)}$$

$$\boldsymbol{y} = \boldsymbol{b}\sin(\boldsymbol{\omega t}) \qquad \textbf{(2)}$$

Equation (1) implies $\boldsymbol{x/a} = \sin(\boldsymbol{\omega t} + \phi)$

$$= \sin \boldsymbol{\omega t cos\phi} + cos\boldsymbol{\omega t}\ \boldsymbol{sin\phi}$$

$$= \sin \boldsymbol{\omega t cos\phi} + \sqrt{(1 - sin^2\boldsymbol{\omega t})}\ \boldsymbol{sin\phi}$$

$$= \mathbf{y/b}\ \boldsymbol{cos\phi} + \sqrt{(1 - (\mathrm{y/b})^2)}\ \boldsymbol{sin\phi} \qquad \textbf{(3)}$$

$$\frac{\boldsymbol{x}}{\boldsymbol{a}} - \frac{\boldsymbol{y}}{\boldsymbol{b}}\boldsymbol{cos\phi} = \sqrt{(1 - (\mathrm{y/b})^2)}\boldsymbol{sin\phi}$$

Squaring both sides we have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\boldsymbol{cos}^2\,\phi - 2.\frac{\boldsymbol{x}}{\boldsymbol{a}}.\ \frac{\boldsymbol{y}}{\boldsymbol{b}}\boldsymbol{cos\phi} = (1 - y^2/b^2)\boldsymbol{sin}^2\,\phi$$

$$or, \frac{x^2}{a^2} + \frac{y^2}{b^2}\boldsymbol{cos}^2\,\phi + \frac{y^2}{b^2}\boldsymbol{sin}^2\,\phi - 2.\frac{\boldsymbol{x}}{\boldsymbol{a}}.\ \frac{\boldsymbol{y}}{\boldsymbol{b}}\boldsymbol{cos\phi} = \boldsymbol{sin}^2\,\phi$$

$$or, \frac{x^2}{a^2} + \frac{y^2}{b^2} - 2.\frac{\boldsymbol{xy}}{\boldsymbol{ab}}\boldsymbol{cos\phi} = \boldsymbol{sin}^2\,\phi \qquad \text{(iv)}$$

which represents the general equation of a conic whose shape will depend upon the value of the phase difference ϕ between two vibrations.

Case I

If $\phi = 0, 2\,\pi, 4\pi\ \ldots\ldots\ldots.. = 2n\pi$

where n = 0,1,2……………… $\boldsymbol{sin\phi} = 0,\ \boldsymbol{cos\phi} = 1$

Equation (iv) implies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2.\frac{\boldsymbol{xy}}{\boldsymbol{ab}} = 0$$

$$or, \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

$$or, \frac{x}{a} - \frac{y}{b} = 0$$

$$or, y = \frac{bx}{a}$$

$y = \frac{bx}{a}$ is the equation of a straight line passing through the origin and inclined to the direction of first motion . i. e. the x-axis at an angle $tan^{-1}\frac{b}{a}$, the resultant amplitude **is** $\sqrt{a^2 + b^2}$. If in addition a = b, then the line will be inclined at an angle of 45°.

Case II

$\phi = \pi/4$ radian, $sin\phi = cos\phi = 1/\sqrt{2}$

(iv) implies $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \sqrt{2}\frac{xy}{ab} = 1/2$

This represents the equation of an oblique ellipse inscribed a rectangle whose length parallel to the x-axis in 2a and breadth 2b. The ellipse touches the rectangle at point$(\pm a, \pm b/\sqrt{2})$ and $(\pm a/\sqrt{2}, \pm b)$.

Case III

$\phi = \pi/2$ radian, $sin\phi = 1$ & $cos\phi = 0$

Equation (iv) implies $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

This represents a symmetrical ellipse whose centre coincides with the origin. The semi-major and semi-minor axes of length 2a and 2b respectively coincide with the co-ordinate axes. If a = b the amplitudes of the two vibrations are equal.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$or, x^2 + y^2 = a^2$$

which is the equation of a circle of radius a. The locus of the particle becomes a circle.

Case IV

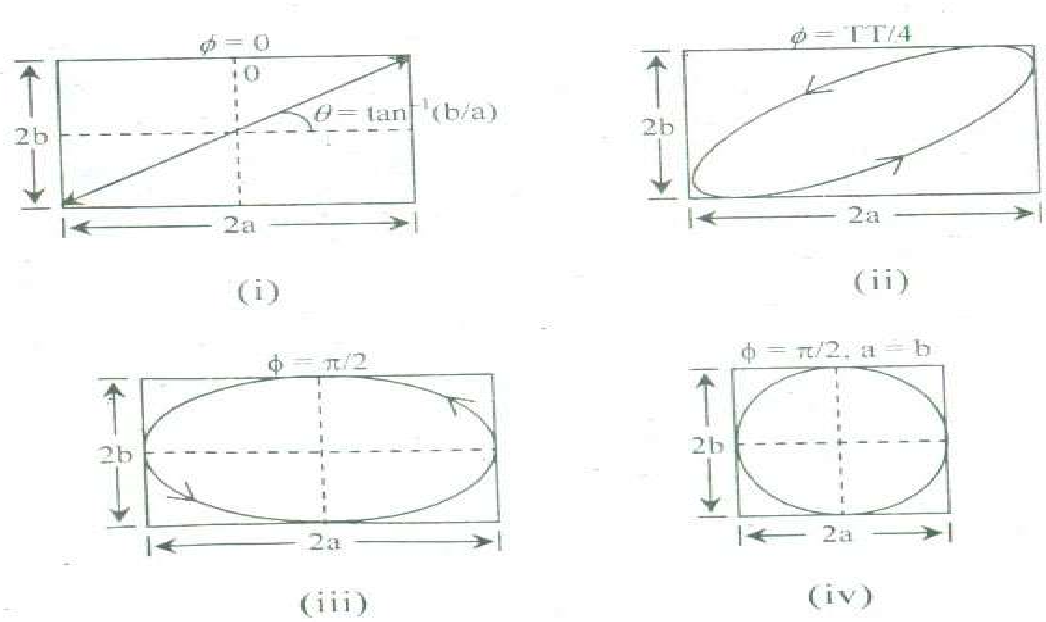
If $\phi = \frac{3\pi}{4} radian$

In that case, $sin\frac{3\pi}{4} = 1/\sqrt{2},\ \boldsymbol{cos}\frac{3\pi}{4} = -1/\sqrt{2}$

Equation (iv) implies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \sqrt{2}.\frac{\boldsymbol{xy}}{\boldsymbol{ab}} = 1/2$$

This represents again the equation of oblique ellipse with the axes rotated by $\frac{\pi}{2}$ with respect to that is case II.



Case V

If $\phi = \pi\ \ radian\ \ then\ \ \boldsymbol{sin\phi} = 0,\ \boldsymbol{cos\phi} = -1$

Equation (iv) implies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2.\frac{xy}{ab} = 0$$

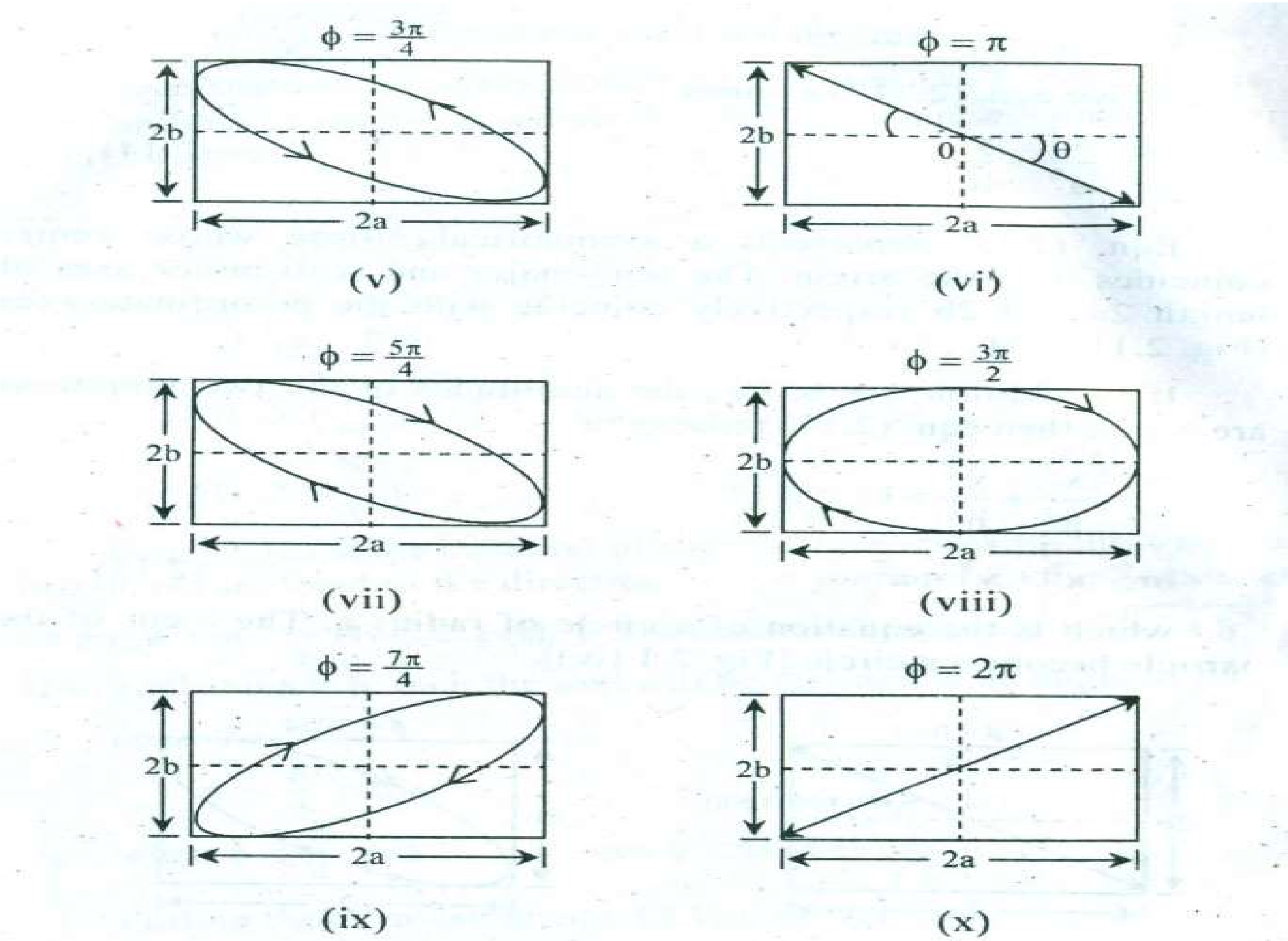
$$or, \left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

$$or, \frac{x}{a} + \frac{y}{b} = 0$$

$$or, y = -\frac{bx}{a}$$

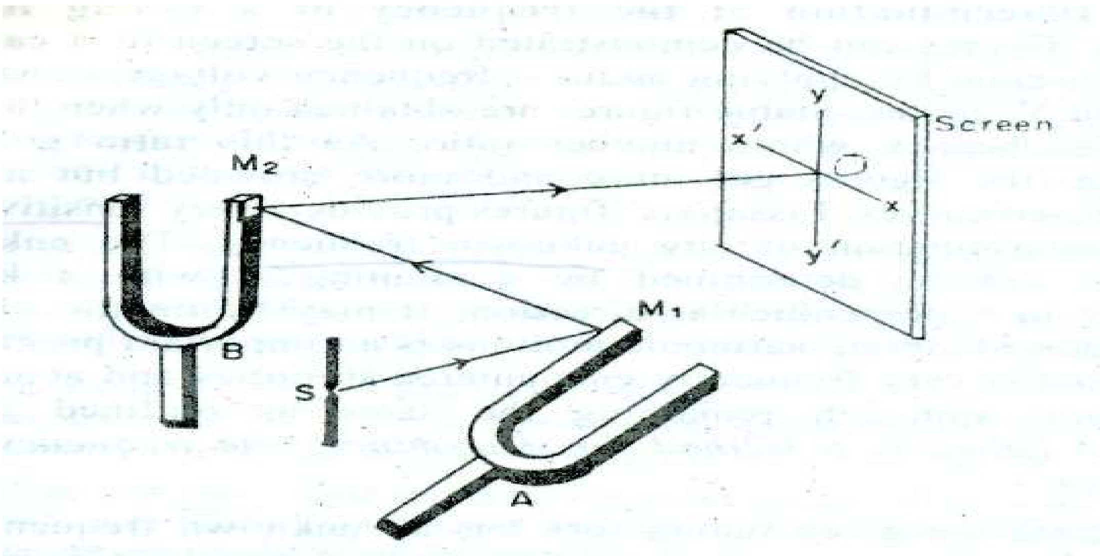
$\boldsymbol{y = -\frac{bx}{a}}$ is the equation of a straight line passing through the origin and with negative slope. This line is inclined with negative x direction at an angle of $tan^{-1}\frac{b}{a}$**,** the resultant amplitude **is** $\sqrt{a^2+b^2}$ and If a = b, then the line will be inclined at an angle of 45°.

when $\phi = \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ *respectively the shape of the curve will be as figure and* 2π will be straight line.



What is Lissajous figure? How can we produce there?

The composition of two SHM is mutually perpendicular directions give rise an elliptical path. The shape of the curve will depend upon the phase difference ϕ and the ratio of the frequencies of the component vibrations. These figures of such composition of two mutually perpendicular SHM was first invented by Lissajous by reflecting a beam of light from two mirrors intern attached to two forks vibrating at right angles to one another is known as Lissajous figures.



The spot of light moves along $x\acute{x}$ when the tuning fork A vibrates and along $y\acute{y}$ when only the tuning fork B vibrates. When both the forks A & B vibrates at right angles to one another the resulting motion of the beam of the light will trace out a figure of eight on the screen and depends on the phase difference of the vibrations and the ratio of the frequencies of the forks.

Uses:

(i)To determine the ratio of the time periods.

(ii) To determine the frequency of a tuning fork.

How can we determine the frequency of a tuning fork using Lissajous figure?

Suppose the tuning fork A has an unknown frequency n_1 and B be another tuning fork of frequency n which is nearly same as that of A. If the two tuning forks are set into vibration in perpendicular plane the Lissajous figure are obtained on the screen. As the tuning forks differ slightly in frequency the phase will differ with the two changes with time. As a result the shape of the Lissajous figures changes continuously with the phase changing from 0 to 2π. Suppose the complete cycle of changes take place in t seconds then the difference in frequencies of A & B = 1/t. therefore the frequency of A = n $\pm$ 1/t . Now attach a tuning wax to the tuning fork A. the experiment is repeated and the time taken to complete one cycle of operation is noted. Let it be t_1, If t_1 > t then $n_1 = n - \frac{1}{t}$ and if t_1 < t then $n_1 = n + \frac{1}{t}$. Thus the frequency of unknown tuning fork can be determined.

Determination of unknown frequency

Let f_1 be the frequency of an unknown tuning fork and f_2 be the frequency of known tuning fork. Let the no. of beat produced per second equal to N.

$$\therefore \boldsymbol{N} = f_1 \sim f_2$$

Now the unknown frequency f_1 may be greater or smaller than the known frequency. So the unknown frequency

$$f_1 = f_2 \pm N$$

- The no. of beats produced per second is equal to the difference of frequencies of the sources.
- If the no. of beats increases then the unknown frequency Will be smaller than the known frequency & the no. of beats decreases then the unknown frequency Will be greater than the known frequency.
- The frequency of the tuning fork is more than the other when no. of beats increases on decreasing weight of its prong.
- The frequency of the tuning fork is less than the other when no. of beats decreases on decreasing weight of its prong.

Ex. 2.2 In an experiment to obtain Lissajous figure one tuning fork is of frequency 256 Hz and a circular figure occurs after every ten seconds. What deduction may be made about the frequency of the other tuning fork?

Solution

Frequency of B = 256Hz

Time of one complete cycle = 10 seconds

Difference in frequencies = 1/t = 1/10 = 0.1 Hz

So the possible frequency of A is

Either 256 + 0.1 = 256.1 Hz

Or, 256 – 0.1 = 255.9 Hz.

Ex. 2.3 Two tuning forks A & B are used to produce Lissajous figures. The frequency of A is slightly greater than that of B is 200Hz. It is found that the figure completes its cycle in 5 seconds. What is the frequency of B?

Solution

Frequency of A = 200 Hz

Time of one complete cycle = 5 seconds

Difference in frequencies = 1/t = 1/5 = 0.2 Hz

Since the frequency of A is greater than the frequency of B.

The frequency of B = 200 – 0.2 = 199.8Hz.

Ex. 2.4 Two tuning forks A & B are of nearly equal frequencies. The frequency of A is 256 Hz.. When the two tuning forks are used to obtain Lissajous' figures, it is found that the complete cycle of changes take place in 20 seconds. When the tuning fork B is loaded with a little wax, the Time taken for one complete cycle of the change is 10 seconds. Calculate the original frequency of B.

Solution

Frequency of A = 256Hz

Time of one complete cycle = 20 seconds

Difference in frequencies = 1/t = 1/20 = 0.05 Hz

So the possible frequency of B is

Either 256 + 0.05 = 256.05 Hz

Or, 256 – 0.05 =255.95 Hz.

After loading time for a complete cycle is 10 seconds. So the time decreases. Let the frequency of B is 256.05 Hz. After loading the frequency of B will be lowered and its difference with the frequency of A becomes less. Therefore the time taken for complete change of cycle be more than 20 seconds.

Hence the frequency of B cannot be 256.05 Hz.

Suppose the frequency of B = 255.95 Hz.

After loading then difference with the frequency of A is increased. The cycle of change will take place in less time.

Hence, the original frequency of B = 255.95 Hz.