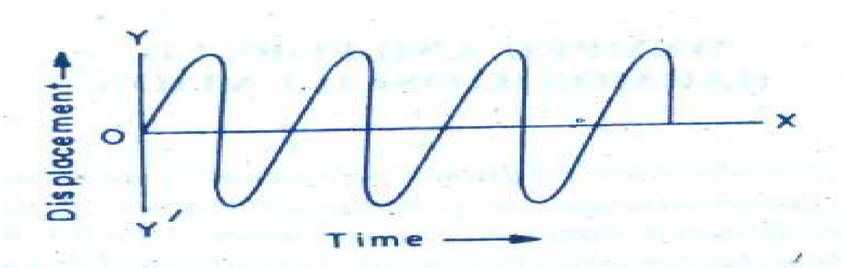
Damped and Forced Harmonic Oscillation

When a simple Harmonic oscillator vibrates in a resisting medium (like air, oil etc.) then the energy is dissipated in each vibration and the amplitude of vibration is decreasing progressively with time. The force resists the vibration is known as damping force.



Thus, a body executing SH oscillation in a damping medium there exists two opposite forces.

1. The restraining force acting on the body which is proportional to the displacement of the body and acts in a direction opposite to the displacement. This force is *-ay* where a is the force constant.
2. A resistive force which is proportional to the velocity of the oscillating body. The resistive force can be written as

$$F = -bv = -b\frac{dy}{dt}$$

Where b is called the damping coefficient of the medium. The $(-)$ ve sign signifies a restraining influence on the vibration of the particle.

- The differential equation of motion of a body executing damped harmonic oscillation can be written as,

$$m\frac{d^2y}{dt^2} = -ay - b\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{a}{m}y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = 0 \qquad (1)$$

where, $2\lambda = \frac{b}{m}$ & $\omega^2 = \frac{a}{m}$

which is the differential equation of a damped harmonic oscillation.

Solution: Let $y = Ae^{kt}$ be the trial solution.

Now,

$\frac{dy}{dt} = kAe^{kt}$, $\frac{d^2y}{dt^2} = k^2Ae^{kt}$

Equation (1) implies $k^2Ae^{kt} + 2\lambda\, kAe^{kt} + \omega^2Ae^{kt} = 0$

$$or, \quad k^2 + 2\lambda k + \omega^2 = 0$$

$$or, \quad k = \frac{-2\lambda \pm \sqrt{(2\lambda)^2 - 4.1.\omega^2}}{2.1}$$

So, $\quad k = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

The general solution is $y = A_1e^{(-\lambda+\sqrt{\lambda^2-\omega^2})t} + A_2e^{(-\lambda-\sqrt{\lambda^2-\omega^2})t}$ **(2)**

where A_1 & A_2 are arbitrary constant.

Now, differentiating equation (2) with respect to t

$$\frac{dy}{dt} = \left(-\lambda + \sqrt{\lambda^2 - \omega^2}\right)A_1e^{(-\lambda+\sqrt{\lambda^2-\omega^2})t} + \left(-\lambda - \sqrt{\lambda^2 - \omega^2}\right)A_2e^{(-\lambda-\sqrt{\lambda^2-\omega^2})t}$$

(3)

Let, the maximum value of the displacement y be $y_{max} = a_0$ at time $t = 0$

Equation (2) implies $y_{max} = a_0 = A_1 + A_2$ **(4)**

Again, the velocity is zero at maximum displacement $\frac{dy}{dt} = 0$ at time $t = 0$

Equation (3) implies $(-\lambda + \sqrt{\lambda^2 - \omega^2})A_1 + (-\lambda - \sqrt{\lambda^2 - \omega^2})A_2 = 0$

$or, -\lambda(A_1 + A_2) + \sqrt{\lambda^2 - \omega^2}(A_1 - A_2) = 0$

$or, \sqrt{\lambda^2 - \omega^2}(A_1 - A_2) = \lambda a_0$

$or,\ A_1 - A_2 = \frac{\lambda a_0}{\sqrt{\lambda^2-\omega^2}}$ **(5)**

$(4) + (5) \Rightarrow 2A_1 = a_0 + \frac{\lambda a_0}{\sqrt{\lambda^2-\omega^2}}$

$$\Rightarrow A_1 = \frac{1}{2}a_0\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right)$$

Equation (4) ⇒ $A_2 = a_0 - \frac{1}{2}a_0\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right) = \frac{1}{2}a_0\left(1 - \frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right)$

Hence, we have, from equation (2)

$$y = \frac{1}{2}a_0 e^{-\lambda t}\{\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right)e^{\sqrt{\lambda^2-\omega^2}\,t} + \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right)e^{-\sqrt{\lambda^2-\omega^2}\,t}\}$$

There are three cases

Case I $when\ (\lambda^2 > \omega^2)$: damping is large.$\lambda^2 - \omega^2$ is $(+)$ve. Hence, the displacement decreases exponentially with time. There exists no oscillation and the motion are called overdamped.

Example: a pendulum oscillating in a viscous fluid like oil.

Case II: When $\lambda^2 = \omega^2$, $\sqrt{\lambda^2-\omega^2} = 0$ then there exists no solution. Let $\sqrt{\lambda^2-\omega^2} = h$ where h → 0

Equation (2) ⇒ ∴ $y = A_1 e^{(-\lambda+h)t} + A_2 e^{(-\lambda-h)t} = e^{-\lambda t}(A_1 e^{ht} + A_2 e^{-ht})$

$$\text{y} = e^{-\lambda}\ \{A_1\left(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \cdots\right) + A_2\left(1 - ht + \frac{h^2t^2}{2!} - \frac{h^3t^3}{3!} + \cdots\right)$$

$$= e^{-\lambda t}\{A_1(1+ht) + A_2(1-ht)\}$$

Let $A_1 + A_2 = M, h\ (A_1 - A_2) = N$

$\therefore y = e^{-\lambda}\ (M + Nt)$ **(6)**

Recall $y_{max} = a_0$, $\frac{dy}{dt} = 0$ at t = 0

∴ $y_{max} = a_0 = M$

$\frac{dy}{dt} = e^{-\lambda t}.N + (M+Nt)(-\lambda)e^{-\lambda t}$ **(7)**

For $\frac{dy}{dt} = 0\ \ at\ t = 0$

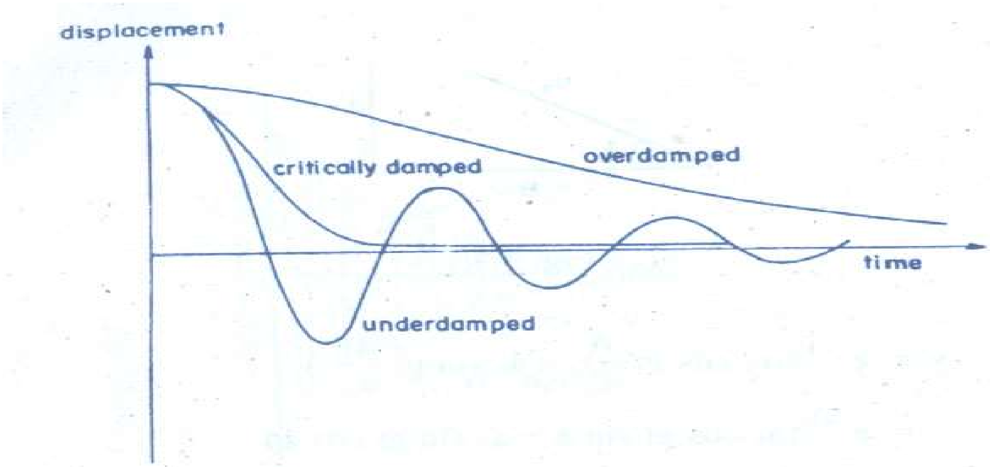
Equation **(7)** implies 0= $N + (M+0)(-\lambda).1$

=>N $= \lambda M = \lambda a_0$

∴(A₁−A_2) $h = \lambda a_0$ **(8)**

y $= e^{-\lambda t}(a_0 + \lambda a_0 t)$

∴ y $= a_0 e^{-\lambda t}(1 + \lambda t)$



Case III: when $\lambda^2 < \omega^2$ then $y = a_0 e^{-\lambda t} \sin(gt + \varphi)$

$\sqrt{\lambda^2 - \omega^2}$ is clearly imaginary say equal to ig when $i = \sqrt{-1}$ and $g = \sqrt{\omega^2 - \lambda^2}$ is a real quantity. From equation (2) we can write

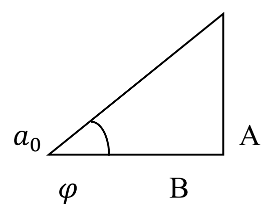
$y = A_1 e^{(-\lambda + ig)t} + A_2 e^{(-\lambda + ig)t}$

or, $y = e^{-\lambda t} A_1 e^{igt} + A_2 e^{-igt}$

$= e^{-\lambda t}\{A_1(cosgt + i\, singt) + A_2(Cosgt - isingt)\}$

$= e^{-\lambda t}\{(A_1 + A_2)cosgt + i(A_1 - A_2)singt\}$

Putting $A_1 + A_2 = A$ & $i(A_1 - A_2) = B$,



We have,

fig-1

y = $e^{-\lambda}$ $(Acosgt + B\, Singt)$ if A, B & a_0 are related as shown in fig-1,

then $y = e^{-\lambda t}(a_0\, cosgt \frac{A}{a_0} + a_0 singt\, \frac{B}{a_0})$

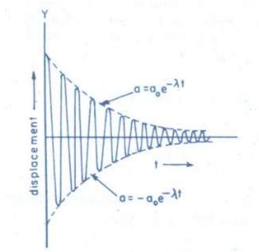
$= e^{-\lambda t}(a_0 cosgtsin\varphi + a_0 singtcos\varphi)$

$= e^{-\lambda}\ a_0\ sin(gt + \varphi)$

which implies $y = e^{-\lambda t} a_0\ sin(gt + \varphi)$

which is the equation of damped harmonic oscillator with amplitude $a_0 e^{-\lambda t}$ and frequency

$\frac{g}{2\pi} = \frac{\sqrt{\ \omega^2 - \lambda^2}}{2\pi}$


Y
displacement
$a = a_0 e^{-\lambda t}$
t
$a = -a_0 e^{-\lambda t}$

Power dissipation in damped harmonic oscillation

Show that the average loss of energy $= 2\lambda E$

Solution

The oscillation of damped harmonic oscillator is given by

$y = a_0 e^{-\lambda} \sin(gt + \varphi)$

So its velocity at a given instant is

$\frac{dy}{dt} = a_0 e^{-\lambda t}(-\lambda \sin(gt + \varphi) + g\cos(gt + \varphi))$

& the kinetic energy of the oscillation of the particles at the instant t

$\frac{1}{2} m \left(\frac{dy}{dt}\right)^2 = \frac{1}{2} m a_0 (a_0)^2 e^{-2\lambda t} (-\lambda \sin(gt + \varphi) + g\cos(gt + \varphi))^2$

$$= \frac{1}{2} m (a_0)^2 e^{-2\lambda} (\lambda^2 \sin^2(gt + \varphi) + g^2 \cos^2(gt + \varphi) - 2\lambda g \sin(gt + \varphi)\cos(gt + \varphi))$$

The average value of the kinetic energy of the damped harmonic oscillator over a complete cycle at the given instant t

$$= \frac{1}{2} m a_0{}^2 e^{-2\lambda t} \left(\frac{1}{2}\lambda^2 + \frac{1}{2} g^2 - 0\right)$$

So the kinetic energy $= \frac{1}{4} m a_0{}^2 e^{-2\lambda} g^2$

[λ^2 **is neglected compared with** g^2]

So, K.E.= $\frac{1}{4} m a_0{}^2 e^{-2\lambda t} g^2$

The potential energy of the oscillator at the given instant t when the displacement y is

$= \frac{1}{2} m\omega^2 y^2$

$= \frac{1}{2} m\omega^2 \left(a_0 e^{-\lambda t} \sin(gt + \varphi)\right)^2$

$= \frac{1}{2} m a_0{}^2 e^{-2\lambda t} \omega^2 \sin^2(gt + \varphi)$

The average value of the potential energy of the DHO over a complete cycle at the given instant t

$\frac{1}{2} m{a_0}^2 \, e^{-2\lambda t} \frac{1}{2} \omega^2$ [since, $\text{Avg}\{\sin^2(gt+\varphi)\} = \frac{1}{2}$]

$= \frac{1}{4} m{a_0}^2 \ \omega^2 \, e^{-2\lambda t}$ [since g$\approx \ \omega$]

$= \frac{1}{4} m{a_0}^2 \ g^2 \, e^{-2\lambda t}$

The average total energy of the oscillator is given by

E = average P.E. + average K.E.

$$\frac{1}{4} m{a_0}^2 \ g^2 \ e^{-2\lambda} \quad + \frac{1}{4} \ m(a_0)^2 \mathrm{g}^2 e^{-2\lambda t}$$

$$= \frac{1}{2} m{a_0}^2 \ g^2 \ e^{-2\lambda t} = E_0 e^{-2\lambda t}$$

where $\frac{1}{2} m{a_0}^2 \ g^2 = E_0 =$ The average total energy of the un damped oscillator

The average power dissipation (p)= the loss of energy = $-\frac{dE}{dt} = -\frac{1}{2} m{a_0}^2 \ g^2 \ e^{-2\lambda t} = 2\lambda E$

Quality factor: The Quality factor of a harmonic oscillator is defined as 2π times the ratio between the energy stored and the energy lost per period. Thus,

$$Q = 2\pi \frac{energy\ stored}{energy\ lost \text{ per period}} = 2\pi \frac{E}{2\lambda ET}$$

$$= \frac{Eg}{2\lambda} = \frac{g}{2\lambda}$$

$$so,\ Q = \frac{\omega}{2\lambda}$$

Again, $Q = \sqrt{\frac{a}{m}}\frac{m}{b} = \frac{\sqrt{am}}{b}$

$$when\ b \to 0, Q \to \alpha$$

$g = \frac{2\pi}{T}$ is the frequency of DHO

Incase of low damping g $\approx\ \omega$

$\omega = \sqrt{\frac{a}{m}}$, frequency

$$2\lambda = \frac{b}{m}$$

So, for lower value of damping the higher value of Q.

Ex. 3.4 A particle of mass 3gm is subjected to an elastic force of 48dyne/cm a damping force of 12 dyne/cm. If the motion is oscillatory find its period.

Solution

$$\omega = \sqrt{\frac{a}{m}} = \sqrt{\frac{\frac{48\text{dyne}}{\text{cm}}}{3g}} = 4/sec$$

$$\&\ 2\lambda = \frac{b}{m}$$

or, $\lambda = \frac{b}{2m} = \frac{12}{2\times 3} = 2/sec$

Since, $\omega > \lambda$ the motion is oscillatory.

So, the frequency of the oscillatory motion is $\frac{g}{2\pi} = \frac{\sqrt{\omega^2 - \lambda^2}}{2\pi}$

Hence the period of the oscillatory motion is

$$\frac{1}{\frac{g}{2\pi}} = \frac{2\pi}{\sqrt{\omega^2 - \lambda^2}} = \frac{2\pi}{\sqrt{4^2 - 2^2}} = \frac{2\pi}{\sqrt{12}} = 1.81 \text{ sec}(approx.)$$

Ex. 3.5 Suppose a tuning fork in air has a frequency $\frac{g}{2\pi} = 200cps$ and its oscillation die away to $\frac{1}{e}$ of its former amplitude is one second. Show that the reduction in frequency by air damping is exceeding small.

Solution

For damping vibration $\frac{g}{2\pi} = \frac{\sqrt{\omega^2 - \lambda^2}}{2\pi}$

Let the amplitude at time t is $y = Ae^{-\lambda}$ (i)

Then the amplitude after one second $\acute{y} = \frac{y}{e} = Ae^{-\lambda(t+1)}$ (ii)

Dividing equation (ii) by (i) we have

$\frac{1}{e} = e^{-\lambda}$

or , $e^{-1} = e^{-\lambda}$

So, $\lambda = 1$

Hence, $\frac{g}{2\pi} = 200 = \frac{\sqrt{\omega^2 - 1}}{2\pi}$

Therefore the value of $\omega = \sqrt{\{(2\pi \times 200)^2 + 1\}}$

or, $\frac{\omega}{2\pi} = \sqrt{\left\{(200)^2 + \frac{1}{4\pi^2}\right\}}$

Since, $\frac{1}{4\pi^2}$ is negligible compared to $(200)^2$, damping due to air has only negligible effect of of the frequency of the tuning fork.

Write down the equation of motion of forced vibration and solve it. Also find the expression for maximum amplitude and quality factor.

Consider the periodic force

$F = F_0 \text{sinpt}$

is applied on a damped harmonic oscillator then the equation of motion for forced vibration can be written as

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - ay + F$$

or, $m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = F_0\text{sinpt}$

or, $\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{a}{m}y = \frac{F_0\text{sinpt}}{m}$

or, $\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = f_0\text{sinpt}$ (1)

where, $2\lambda = \frac{b}{m}, \omega^2 = \frac{a}{m}$ & $f_0 = \frac{F_0}{m}$

Let the particular solution of equation (1) is

$y = A\sin(pt - \theta)$

or, $\frac{dy}{dt} = pA\cos(pt - \theta)$

or, $\frac{d^2y}{dt^2} = -Ap^2\sin(pt - \theta)$

or, $\frac{d^2y}{dt^2} = -p^2 y$

So, equation (1) implies $-Ap^2\sin(pt - \theta) + 2\lambda Ap\cos(pt - \theta) + \omega^2 A\sin(pt - \theta)$

$= f_0\sin((pt - \theta) + \theta)$

$= f_0\sin(pt - \theta)\cos\theta + f_0\sin\theta\cos(pt - \theta)$

Now by separating the coefficient of $\sin(pt - \theta)$ & $\cos(pt - \theta)$ we have,

$A(\omega^2 - p^2) = f_0\cos\theta$ (2)

$2\lambda Ap = f_0\sin\theta$ (3)

$(2)^2 + (3)^2$ implies

$A^2(\omega^2 - p^2)^2 + 4\lambda^2 A^2 p^2 = {f_0}^2(\sin^2\theta + \cos^2\theta)$

or, $A^2(\omega^2 - p^2)^2 + 4\lambda^2 A^2 p^2 = {f_0}^2$

or, $A^2((\omega^2 - p^2)^2 + 4\lambda^2 p^2) = {f_0}^2$

or, $A^2 = \frac{{f_0}^2}{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}$

Hence, the amplitude of the forced oscillator is $A = \sqrt{\frac{{f_0}^2}{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}}$

$(3) \div (2)$ implies $\frac{f_0\sin\theta}{f_0\cos\theta} = \frac{2\lambda Ap}{A(\omega^2 - p^2)}$

so, $\tan\theta = \frac{2\lambda p}{\omega^2 - p^2}$

or, $\theta = \tan^{-1}\frac{2\lambda p}{\omega^2 - p^2}$

which is the phase difference between the driven or forced oscillator and the applied force.

Hence the particular solution is $y = \sqrt{\frac{{f_0}^2}{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}} \times sin\left(pt - \tan^{-1}\frac{2\lambda p}{\omega^2 - p^2}\right)$ represents a SHM of frequency $\frac{p}{2\pi}$.

Again the complementary function of (1) which is solution of $\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = 0$ is (we know)

$$y = a_0 e^{-\lambda} \sin(gt + \varphi)$$

Thus, the complete solution of (1) will be

$$y = a_0 e^{-\lambda t} \sin(gt + \varphi) + A \sin(pt - \theta)$$

Maximum Amplitude: $A^2 = \frac{{f_0}^2}{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}$

The amplitude will be maximum if $(\omega^2 - p^2)^2 + 4\lambda^2 p^2$ has its minimum value.

$$\frac{d}{dt}\{(\omega^2 - p^2)^2 + 4\lambda^2 p^2\} = 0$$

or , $-2(\omega^2 - p^2)2p + 4\lambda^2 2p = 0$

or, $\omega^2 - p^2 = 2\lambda^2$

or, $p^2 = \omega^2 - 2\lambda^2$

So, $p = \sqrt{\omega^2 - 2\lambda^2}$

Hence, the expression for Maximum Amplitude $A_{max} = \sqrt{\frac{{f_0}^2}{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}}$

or, $A_{max} = \sqrt{\frac{{f_0}^2}{(\omega^2 - (\omega^2 - 2\lambda^2))^2 + 4\lambda^2(\omega^2 - 2\lambda^2)}}$

$$= \frac{f_0}{\sqrt{4\lambda^2\omega^2 - 4\lambda^4}} = \frac{f_0}{\sqrt{4\lambda^2(\omega^2 - \lambda^2)}} = \frac{f_0}{2\lambda\sqrt{(\omega^2 - \lambda^2)}}$$

Quality factor: The ratio of the amplitude of the oscillator when the driving frequency is very small is called the **Quality factor** of the oscillator and is denoted by Q

Hence,

$Q = \frac{f_0/2\lambda\omega}{f_0/\omega^2} = \frac{\omega}{2\lambda}$ & the relaxation time is $\tau = \frac{Q}{\omega}$ where $Q = \frac{A_{max}}{A} = \frac{f_0/2\lambda\omega}{f_0/\omega^2}$ & $\omega = 2\pi n$

H.W. 3.6, 3.7, 3.8