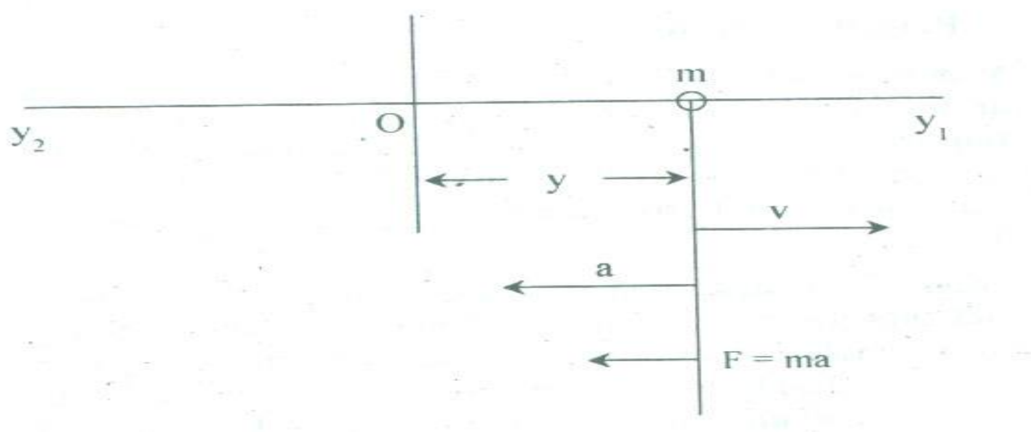
Physics

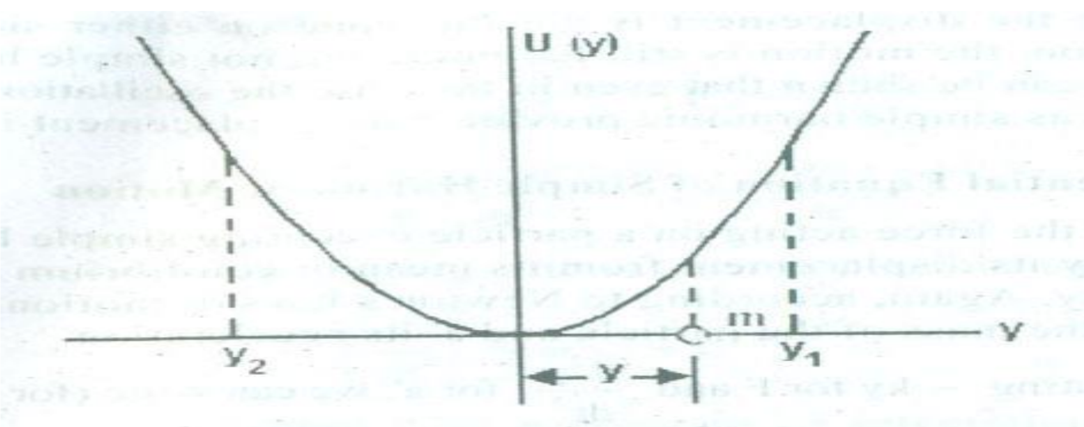
Simple Harmonic Motion

Periodic motion: A motion which repeats over and over after a regular interval of time is called periodic motion.

Simple Harmonic Motion (SHM): If the motion of a particle in such a way that it's acceleration is proportional to it's displacement and always directed towards its equilibrium position is called simple harmonic motion.



Differential equation of Simple Harmonic Motion: If the force acting on a particle executing SHM & y its displacement from its mean position then,



$$F = -ky \qquad \textbf{(1)}$$

According to Newton's Law of motion

$$F = ma = m\frac{d^2y}{dt^2} \qquad \textbf{(2)}$$

From equation (1) & (2)

$$-ky = m\frac{d^2y}{dt^2}$$

$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$ which is the required differential equation of SHM

Again,

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y = -\omega^2 y \qquad (3)$$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular velocity of the particle.

To find the general Solution. We have

$$2\frac{dy}{dt}.\frac{d^2y}{dt^2} = -\omega^2 y.\ 2\frac{dy}{dt}$$

Integrating with respect to t we have

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + c$$

When $\frac{dy}{dt} = 0$ at y = a, the velocity is zero at maximum displacement.

Or, $0 = -\omega^2 a^2 + c$

Or, $c = a^2\omega^2$

Or, $\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + a^2\omega^2 = \omega^2(a^2 - y^2)$

$So,\ \frac{dy}{dt} = \pm\,\omega\sqrt{a^2 - y^2} \quad = \pm\sqrt{\frac{k}{m}}\sqrt{a^2 - y^2}$

$or, \frac{dy}{\sqrt{a^2-y^2}} = \omega.\ dt$

Integrating again with respect to t we have

$sin^{-1}\frac{y}{a} = \omega t + \varphi$

Or, $y = a\sin(\omega t + \varphi)$ which is the general solution of the differential equation of Simple Harmonic Motion (SHM).

Velocity & Acceleration of a body executing SHM :

The displacement of a particle with SHM is $y = a\, sin\, (\omega t + \varphi)$

1. The velocity of the particle at any instant of t is

$\frac{dy}{dt} = \omega$ a cos $(\omega t + \varphi)$

$= \pm\omega$ a $\sqrt{1 - sin^2(\omega t + \varphi)}$

$= \pm\omega$ a $\sqrt{1 - \frac{y^2}{a^2}}$ [$\sin(\omega t + \varphi) = \frac{y}{a}$]

$= \pm\omega\sqrt{a^2 - y^2}$

So, $\frac{dy}{dt} = \pm\sqrt{\frac{k}{m}}\sqrt{a^2 - y^2}$ (4)

2. The acceleration of the particle at any time t is given by,

$\frac{d^2y}{dt^2} = -\omega^2 a \sin(\omega t + \varphi)$

$= -\omega^2 a \times {}^{y}/_{a}$

$= -\omega^2 y$

$= -\frac{k}{m}$ y (5)

The velocity will be maximum when y = 0 then

$V_{max} = \pm\, \omega a = \pm$ a $\sqrt{\frac{k}{m}}$ when the particle passing it's mean position.

When y is maximum, then acceleration is maximum. That means the particle is at the position of one of the extreme displacements.

Time period, T = $2\pi\sqrt{\frac{m}{k}}$, the particle repeats it's SHM after every $\frac{2\pi}{\omega}$

Frequency, n = $\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, the reciprocal of the time period .

Angular frequency, $\omega = 2\pi n = \frac{2\pi}{T}$, the angular velocity of a Simple Harmonic particle.

Displacement, y = a cos ωt + b sin ωt

Example 1.2 A body is vibrating with SHM of amplitude 15 cm & frequency 4 Hz. Compute (a) the maximum Acceleration & velocity (b) The acceleration & velocity when the displacement is 9 cm.

Solution:

Here, a = 15 cm
n = 4 Hz

(a) $V_{max} = \omega$ a

Again,

$\omega = 2\pi n$

$= 2\times 3.14 \times 4$

= 25.12 rad/s

V_{max} = 25.12×15 cm/s

= 376.8 cm/s

$(\text{Acceleration})_{max} = -\omega^2 a = -(25.12)^2 \times 15 = -9470 \text{ cm}/s^2$

(b) When y = 9 cm.

$V = \omega\sqrt{a^2 - y^2}$

$= 25.12\ \sqrt{(15)^2 - 9^2} = 300 cm/sec$

Acceleration = $-\omega^2 y$

$= -(25.12)^2 \times 9$

= - 5680 cm/sec^2

Example 1.4: The displacement of an oscillating particle at any instant t is given by

$$y = a \cos\omega t + b \sin\omega t$$

Show that it's motion is SHM. If a = 5 cm, b = 12 cm & ω = 4 radian/sec. Calculate

(i) The amplitude (ii) The time period

(iii)The maximum velocity (iv) The maximum acceleration of the particle.

Solution: y = a cos ωt+ b sin ωt

or, $\frac{dy}{dt}$ = - a ω sin ωt + b ω cos ωt

or, $\frac{d^2y}{dt^2}$ = -a$\omega^2 cos\ \omega t - b\omega^2 sin\omega t$

= -ω^2(a cosωt+ b sinωt)

$or, \frac{d^2y}{dt^2} + \omega^2 y = 0$

Hence, the motion is SHM.

(i) Let,

a = A sinα , b = A cosα

y = A sin α cos ωt + A cos α sin ωt

or, y = A sin (ωt + α) which represents a SHM with amplitude A.

(ii) Again $a^2 + b^2 = A^2 sin^2\alpha + A^2 cos^2\alpha = A^2$

or, A = $\sqrt{a^2 + b^2}$

or, A = $\sqrt{5^2 + (12)^2}$

or, A = 13 cm

a = 5 cm

b = 12cm

(iii) $V_{max.} = \omega A = 4\times 13 = 52$ cm/s

(iv) $(Acceleration)_{max} = -\omega^2 A = -(4)^2\times 13 = -208$ cm/s^2

Example 1.7: A Simple harmonic motion is represented by $y = 10sin\left(10t - \frac{\pi}{6}\right)$

Calculate (i) the frequency (ii) the time period, (iii) the maximum displacement , (iv) the maximum velocity (v) the maximum acceleration (iv) displacement, velocity and acceleration at time t = 0 and t = 1second.

Solution:

Here, $y = 10sin\left(10t - \frac{\pi}{6}\right)$ (1)

Comparing equation (1) with $y = a \sin(\omega t+\delta)$ **(2)**

We have,

$$\omega = 2\pi n = 10$$

$$\text{or, } n = \frac{10}{2\pi} = 1.6 \text{ Hz}$$

Time period $T = \frac{1}{n} = \frac{2\pi}{10} = 0.63$ s

Maximum displacement is amplitude y = a = 10 m

Maximum velocity, $V_{max.} = \omega a = 10\times 10 = 100$ m/s

Maximum Acceleration $= -\omega^2 a = -10^2 \times 10 = -1000$ m/s^2

At t = 0, $y = 10 \sin(10\times 0 - \frac{\pi}{6})$

$= 10\times (-\frac{1}{2})$

$= -5$ m

Velocity, $\frac{dy}{dt} = a\,\omega \cos(\omega t+\delta)$

$= 10\times 10 \cos(0 - \frac{\pi}{6})$

$= 100.\cos\frac{\pi}{6}$

$= \frac{100\sqrt{3}}{2}$

Acceleration, $\frac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t+\delta)$

$= -10\times 10^2 \sin(-\frac{\pi}{6})$

$= \frac{1000}{2}$

$= 500$ m/s^2

At t = 1, $y(1) = 10\sin(10\times 1 - \frac{\pi}{6})$

$= 10\sin(\frac{60-3.14}{6})$

$= 10\sin\left(\frac{36.858}{6}\right)$

$= 10\sin 3\pi$

Velocity, $\frac{dy}{dt} = a\omega\cos(10t - \frac{\pi}{6})$

$= a\omega\cos\pi$

$= -100$ m/s

Acceleration, $\frac{d^2y}{dt^2} = -a\omega^2 \sin\left(10t - \frac{\pi}{6}\right)$

$= -a\omega^2 \sin\pi_{(app)} = 0$

Energy of the body Executing SHM: Since there exists acceleration. So, force for executing SHM. Consequently, work is done during the displacement of the particle. Hence, the particle possesses potential energy (P.E.). As the particle have velocity so it possesses kinetic energy (K.E.). So, if no frictional force on the particle the sum of the K.E.+ P.E. will remain constant.

or, E = K + U = constant.

Total potential energy $U = \int_0^y F.dy$

$= \int_0^y m\omega^2 y.dy$

$= m\omega^2 [y^2/2]_0^y$

$= m\,\omega^2\, y^2/2$

$= \frac{1}{2} m\omega^2 a^2 \sin^2(\omega t+\varphi)$

$= \frac{1}{2} k\, a^2 \sin^2(\omega t+\varphi)$

$F = ma$

$= m\frac{d^2y}{dt^2}$

$= m(-\omega^2 y)$

$= -m\omega^2 y$

Energy cannot be (-) ve

Since, $k = \omega^2 m$

& the kinetic energy of the particle is

$K.E. = \frac{1}{2} mv^2$

$= \frac{1}{2} m\left(\frac{dy}{dt}\right)^2$

$= \frac{1}{2} m\{\omega a\cos(\omega t+\varphi)\}^2$

$= \frac{1}{2}$ k $a^2 \cos^2(\omega t+\varphi)$

Total energy E = K+U = $\frac{1}{2}$ k $a^2\cos^2(\omega t+\varphi)$+ $\frac{1}{2}$ k $a^2\sin^2(\omega t+\varphi)$

$= \frac{1}{2}$ ka^2

$= \frac{1}{2}$ m$\omega^2 a^2$

Again,

$\omega = \frac{2\pi}{T}$

Total Energy, T.E. = $\frac{1}{2}$ m $(\frac{2\pi}{T})^2 a^2$

$= \frac{2\pi^2 ma^2}{T^2} \quad = 2\pi^2 mn^2a^2$ [n = $\frac{1}{T}$]

- K.E max. at the equilibrium position but P.E. is zero.
- K.E is zero at the max. displacement where P.E. is max.

Example 1.13. A 0.2 kg mass suspended from a spring with SHM, where T= 3s, amplitude a = 10cm. Find (i) the force constant k of the spring (ii) Find the displacement, Velocity & acceleration at t =1s (iii) Show that P.E.+K.E. = $\frac{1}{2}$ ka^2 at t =1 s.

Solution:

(i) We know, $\omega = \sqrt{\frac{k}{m}}$

or, $\omega^2 = \frac{k}{m}$

or, $(\frac{2\pi}{T})^2 = \frac{k}{m}$

or, k = $\frac{4\pi^2}{T^2}$m

$= \frac{4\pi^2}{3^2} \times 0.2$

Force Constant, k = 0.88 N/m

(ii) Consider upward direction as the positive direction,

Then from

$$y = a \sin \omega t$$

We have

At $t = 0$, $y = 0$

At $t = 1s$, $y = 0.10 \sin\left(\frac{2\pi}{3}\right).1$

$= 0.1 \sin 120^0$

$= 0.0866$ m

Here,

$a = 10$ cm

$= 0.1$ m

$T = 3s$

Velocity, $v = \frac{dy}{dt} = a\omega \cos\omega t = \frac{2\pi}{T} a \cos\frac{2\pi}{T}(1) = \frac{2\pi}{3}(0.1) \cos 120^0 = -0.105$ m/s

Acceleration $= \frac{d^2y}{dt^2} = -\omega^2 a \sin \omega t = -0.38$ m/s^2

$E = P.E. + K.E.$

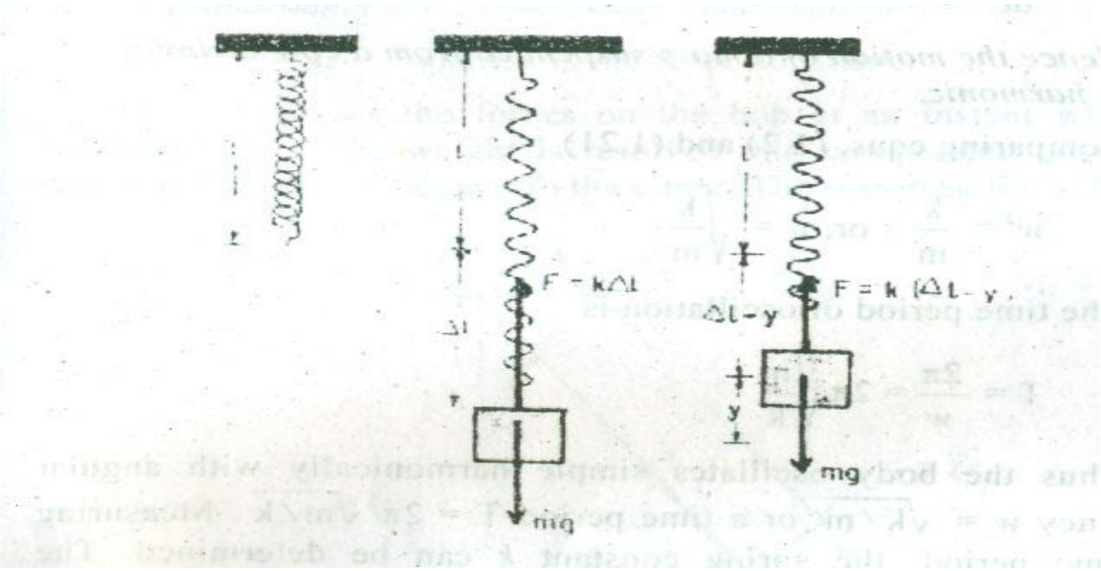
or, $\frac{1}{2} ka^2 = \frac{1}{2} ky^2 + \frac{1}{2} kv^2$

or, $\frac{1}{2}(0.88)(0.1)^2 = \frac{1}{2}(0.88)(0.0866)^2 + \frac{1}{2}(0.88)(0.105)^2$

or, $4.4\times 10^{-3}J = 4.4\times 10^{-3}J$

Show that the spring mass system is Simple Harmonic Motion

Consider the situation as shown in figure. Let at any instant the mass is at B. The distance is AB = y. Let the tension per unit displacement of the spring be k. Force exerted by the Spring = ky.



According to Newton's 2nd law $F = m\frac{d^2y}{dt^2} \;\; \alpha - y$

$$or,\; m\frac{d^2y}{dt^2} = -ky$$

$$\text{or},\; \frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

or, $\frac{d^2y}{dt^2} + \omega^2 y = 0$ which is the equation of SHM.

Time period of oscillation is T = $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Thus the body oscillates simple harmonically with angular frequency $\omega=\sqrt{\frac{k}{m}}$ or time period

$T = 2\pi\sqrt{\frac{m}{k}}$ where k is the spring constant can be found k = mg/x .

Show that the average value of K.E. and P.E. is equal to the half of the total energy

Solution:

The potential energy (P.E.) of the particle at a displacement y is given by

PE = ½ $m\omega^2 y^2$

= ½ mω^2 a^2 sin^2(ωt+φ)

So the average potential energy of the particle over a complete cycle or a whole time period T

$$= \quad 1/T \int_0^T 1/2\ m\omega^2 a^2 sin^2(\omega \mathrm{t} + \varphi)dt$$

$$= \quad 1/T \frac{m\omega^2 a^2}{4} \int_0^T 2\ sin^2(\omega \mathrm{t} + \varphi)dt$$

$$= \quad 1/T \frac{m\omega^2 a^2}{4} \int_0^T (1 - \cos 2(\omega \mathrm{t} + \varphi))dt$$

The average value of both a sine and a cosine function for a complete cycle is zero. Therefore the average potential energy of the particle

$$= \quad 1/T \frac{m\omega^2 a^2}{4} T$$

$$= \quad \frac{m\omega^2 a^2}{4}$$

$$= \quad \frac{ka^2}{4}$$

Again, The kinetic energy of the particle at displacement y is

$$\mathrm{K.E.} = \frac{1}{2}\mathrm{mv}^2$$

$$= \frac{1}{2}\mathrm{m}\left(\frac{dy}{dt}\right)^2$$

$$= \frac{1}{2}\mathrm{m}\ \{\omega \mathrm{a} \cos(\omega \mathrm{t} + \varphi)\}^2$$

$$= \frac{1}{2} m\omega^2 \mathrm{a}^2 \cos^2(\omega \mathrm{t} + \varphi)$$

So the average kinetic energy of the particle over a complete cycle or a whole time period T

$$= \quad 1/T \int_0^T \frac{1}{2}\ m\omega^2 a^2\ cos^2\ (\omega \mathrm{t} + \varphi)\ dt$$

$$= \quad 1/T \frac{m\omega^2 a^2}{4} \int_0^T 2\ cos^2(\omega \mathrm{t} + \varphi)dt$$

$$= \quad 1/T\,\frac{m\omega^2 a^2}{4}\int_0^T (1 + \cos 2(\omega t + \varphi))dt$$

The average value of both a sine and a cosine function for a complete cycle is zero. Therefore the average kinetic energy of the particle

$$= \quad 1/T\,\frac{m\omega^2 a^2}{4}T$$

$$= \quad \frac{m\omega^2 a^2}{4}$$

$$= \quad \frac{ka^2}{4}$$

Thus the average value of potential energy = the average value of kinetic energy of the particle

$$= \quad \frac{m\omega^2 a^2}{4}$$

$$= \quad \frac{ka^2}{4}$$

= half of the total energy.