DIGITAL SIGNAL PROCESSING

THIRD EDITION

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Preface

This book *Digital Signal Processing, 2e* is designed for a one-semester course on the subject for students of engineering at the undergraduate level. The book has its emphasis on ease of comprehension, lucid explanation of concepts, numerous examples, a solved problems approach and simple presentation.

The first four chapters of the book provides an introduction and recap of the topics on signals and systems. Students who have studied Signals and Systems as a core paper may decide to skip these chapters.

A significant highlight of this book is the treatment and coverage of topics on Finite Impulse Response Filters (FIR) and Infinite Impulse Response Filter (IIR). It reckons for students who find the subject tough and provides numerous examples with explanations.

Similarly, the topic Finite World Length Effect has its emphasis on clear concepts and a simple and easy to understand presentation. The coverage of the topic has a prerequisite that the students are familiar with number and decimal systems.

The topic Multirate Signal Processing has been discussed with necessary mathematical treatment. The basic concepts are explained in simple English to facilitate better comprehension.

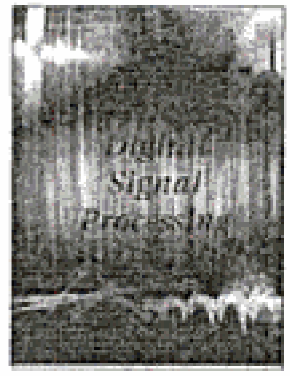
This textbook has a crisp and clear introduction to Estimation Theory starting from Estimation Parameters to Model Estimation.

The field of Digital Signal Processing has its impact on all areas of technology and science. It is of equal importance to industry and academia. In the engineering curriculum, this subject is now offered to students of electronics, electrical, communication, IT and computer science streams. We hope this book will serve as a basic resource for all students of engineering.

We thank the management and staff members of our respective institutions for all their support and help. We thank Mr. P K Madhavan and others of Vijay Nicole for their efficient and tireless efforts in publishing this book.

We welcome all objective criticisms and suggestions on the book.

Dr S Poornachandra
B Sasikala



CHAPTER

1

Introduction to Digital Signal Processing

■ 1.1 WHAT IS DSP?

DSP or Digital Signal Processing, as the term suggests, is the processing of signals by digital means. A signal in this context means a source of information. In general terms, a signal is a stream of information representing anything from stock prices to data from a remote-sensing satellite. The signal here means an electrical signal carried by a wire or telephone line, or perhaps by a radio wave. In many cases, the signal is initially in the form of an analog electrical voltage or current, produced for example by a microphone or some other type of transducer. In some situations the data is already in digital form—such as the output from the readout system of a CD (compact disc) player. An analog signal must be converted into digital (i.e. numerical) form before DSP techniques can be applied. An analog electrical voltage signal, for example, can be digitized using an analog-to-digital converter (ADC). An analog signal on sampling results in a discrete signal followed by quantization and encoding in order to convert the discrete signal to digital signal. This generates a digital output in the form of a binary number whose value represents the electrical voltage input to the device.

Signals need to be processed in a variety of ways. For example, the output signal from a transducer may be contaminated with noise. The electrodes attached to a patient's chest when an electrocardiogram (ECG) is taken, measure tiny electrical voltage changes due to the activity of the heart and other muscles. The signal is often strongly affected due to electrical interference from the mains supply, electromagnetic interference, muscle artifacts, etc. Processing the signal using a filter circuit can remove or at least reduce the unwanted part of the signal. Nowadays, the filtering of signals to improve signal quality or to extract important information is done by DSP techniques rather than by analog electronics.

The development of digital signal processing dates from the 1960s with the use of mainframe digital computers for number-crunching applications such as the Fast Fourier Transform (FFT), which allows the frequency spectrum of a signal to be computed rapidly. These techniques were not widely used earlier because suitable computing equipment was available only in leading universities and other scientific research institutions. The introduction of the microprocessor in the late 1970s and early 1980s made it possible for DSP techniques to be used in a much wider range of applications. However, general-purpose microprocessors such as the Intel x86 family are not ideally suited to the numerically-intensive requirements of DSP, and during the 1980s the increasing importance of DSP led several major electronic manufacturers (such as Texas Instruments, Analog Devices, and Motorola) to develop Digital Signal Processor chips—specialized microprocessors with architectures designed specifically for the types of operations required in digital signal processing. (Note that the acronym DSP can variously mean Digital Signal Processing, the term used for a wide range of techniques for processing signals digitally, or Digital Signal Processor, a specialized type of microprocessor chips). Like a general-purpose microprocessor, a DSP is a programmable device, with its own native instruction code. DSP chips are capable of carrying out millions of floating point operations per second, and like their better-known general-purpose cousins, faster and more powerful versions are continually being introduced.

DSP technology is commonly employed nowadays in devices such as mobile phones, multimedia computers, video recorders, CD players, hard disc drive controllers and modems, and will soon replace analog circuitry in TV sets and telephones. An important application of DSP is in signal compression and decompression. In CD systems, for example, the music recorded on the CD is in a compressed form (to increase storage capacity) and must be decompressed for the recorded signal to be reproduced. Signal compression is used in digital cellular phones to allow a greater number of calls to be handled simultaneously within each local "cell". DSP signal compression technology allows people not only to talk to one another by telephone but also to see one another on the screens of their PCs, using small video cameras mounted on the computer monitors, with only a conventional telephone line linking them together. Although the mathematical theory underlying DSP techniques such as Fast Fourier transform, Wavelet transform, Hilbert transform, Digital filter design and Signal compression can be fairly complex, the numerical operations required to implement these techniques are in fact very simple, consisting mainly of operations that could be done on a cheap four-function calculator. The architecture of a DSP chip is designed to carry out such operations incredibly fast, processing upto tens of millions of samples per second, to provide real-time performance, that is, the ability to process a signal "live" as it is sampled and then output the processed signal, for example, to a loud speaker or video display. All the practical applications DSP mentioned earlier, such as hard disc drives and mobile phones, demand real-time operation.

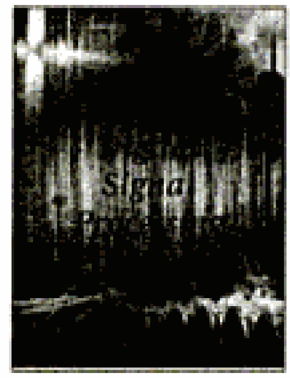
In signal processing, the function of a filter is to remove unwanted parts of the signal, such as random noise, or to extract useful parts of the signal, such as the components lying within a certain frequency range. There are two main kinds of filters, analog and digital. They are quite different in their physical makeup and in their working. An analog filter uses analog electronic circuits made from components such as resistors, capacitors and op amps to produce the required filtering effect. Such filter circuits are widely used in applications such as noise reduction, video signal enhancement, graphic equalizers in *hi-fi* systems, and many other areas. There are well-established standard techniques for designing an analog filter circuit for a given requirement. At all stages, the signal being filtered is an electrical voltage or current, which is the direct analog of the physical quantity (example, a sound or video signal or transducer output) involved.

A digital filter uses a digital processor to perform numerical calculations on sampled values of the signal. The processor may be a general-purpose computer such as a PC, or a specialized DSP (Digital Signal Processor) chip. The analog input signal must first be sampled and digitized using an ADC. The resulting binary numbers, representing successive sampled values of the input signal, are transferred to the processor, which carries out numerical calculations on them. These calculations typically involve multiplying the input values by constants and adding the products together. If necessary, the results of these calculations, which now represent sampled values of the filtered signal, are output through a DAC (digital to analog converter) to convert the signal back to analog form. Note that in a digital filter, the signal is represented by a sequence of numbers, rather than a voltage or current.

The main advantages of digital filters over analog filters are listed below.

1. A digital filter is programmable, that is, its operation is determined by a program stored in the processor's memory. This means the digital filter can easily be changed without affecting the circuitry (hardware). An analog filter can only be changed by redesigning the filter circuit.
2. Digital filters are easily designed, tested and implemented on a general-purpose computer or workstation.
3. The characteristics of analog filter circuits (particularly those containing active components) are subject to drift and are dependent on temperature. Digital filters do not suffer from these problems, and so are extremely stable with respect to both time and temperature.
4. Unlike their analog counterparts, digital filters can handle low frequency signals accurately. As the speed of DSP technology continues to increase, digital filters are being applied to high frequency signals in the RF (radio frequency) domain, which in the past was the exclusive preserve of analog technology.
5. Digital filters are very much more versatile in their ability to process signals in a variety of ways; this includes the ability of some types of digital filter to adapt to changes in the characteristics of the signal.

Fast DSP processors can handle complex combinations of filters in parallel or cascade (series), making the hardware requirements relatively simple and compact in comparison with the equivalent analog circuitry.



CHAPTER

2

Introduction to Signals and Systems

■ 2.1 INTRODUCTION TO MODELING

This book discusses signals and systems related to Engineering. It focuses on the modeling of physical signals and systems by mathematical functions, and the solution of such mathematical functions, when the system is excited by such signals.

2.1.1 Signals

A signal is defined as a function of one or more variables which conveys information. A signal is a physical quantity that varies with time in general, or any other independent variable. It can be dependent on one or more independent variables. Dimension of a signal may be defined based on the number of independent variables.

Any variables which does not convey information is called Noise. Noise is a random phenomenon in which physical parameters are time-variant. Unlike a signal, noise is usually does not carry useful information and is almost always considered undesirable. Some examples include channel noise in communication systems, transformer humming in electrical engineering and moving artifacts in biological systems.

2.1.2 One-dimensional Signal

When a function depends on a single independent variable to represent the signal, it is said to be a one-dimensional signal.

The ECG signal and speech signal shown in Fig. 2.1(a) and 2.1(b) respectively are examples of one-dimensional signals where the independent variable is time. The magnitude of the signals is dependent variable.

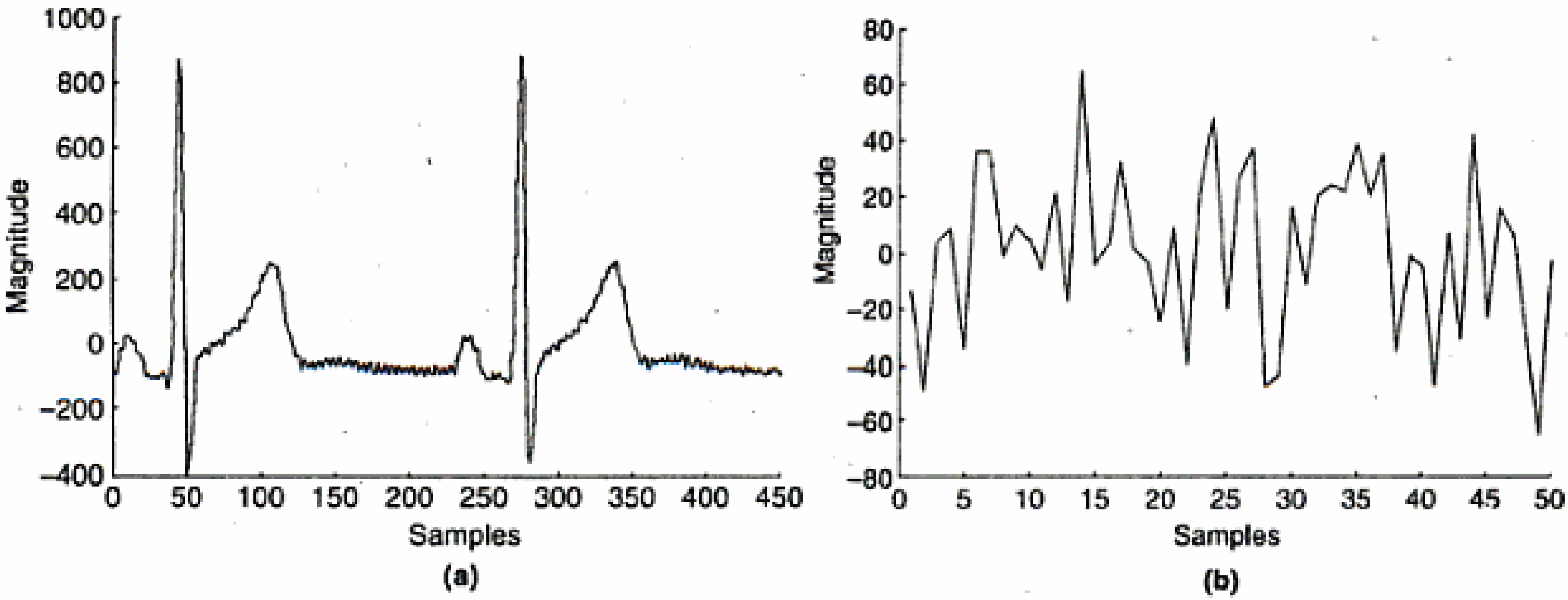


Fig. 2.1 One-dimensional Signal

(a) ECG Signal (b) Speech Signal

2.1.3 Two-dimensional Signal

When a function depends on two independent variables to represent the signal, it is said to be a two-dimensional signal. For example, photograph shown in Fig. 2.2 is an example of two-dimensional signal wherein the two independent variables are the two spatial coordinates which are usually denoted by x and y.

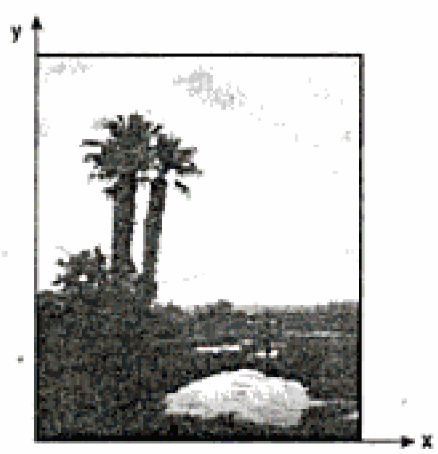


Fig. 2.2 Two-dimensional Photograph

2.1.4 Multi-dimensional Signal

When a function depends on more than one independent variables to represent the signal, it is said to be a multi-dimensional signal. For example, space missile shown in Fig. 2.3 is an example of three-dimensional image.

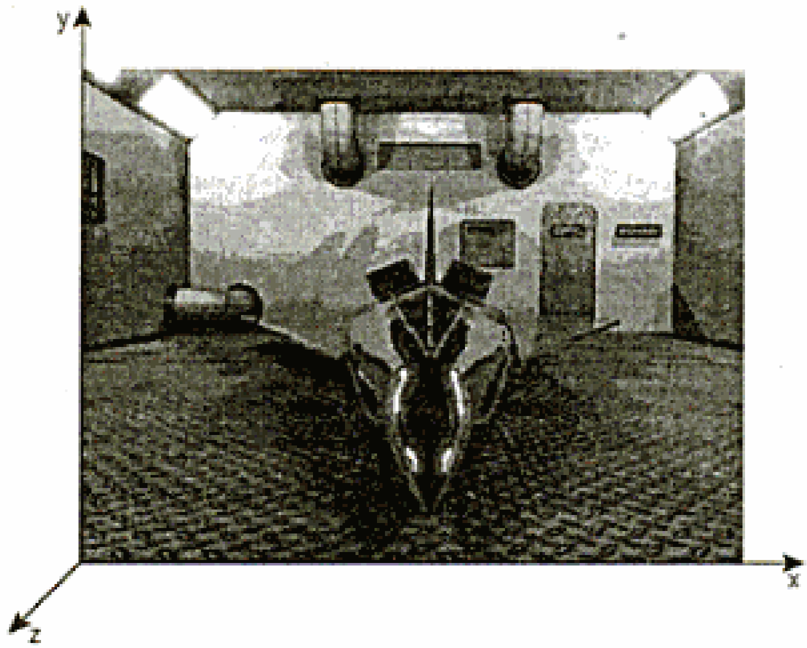


Fig. 2.3 3D-Space Missile

Definition

Input signal A signal that enters a system from an external source is referred to as an input signal. For example, the voltage from a function generator, electrocardiogram from heart, temperature from the human body, etc.

Output signal A signal produced by the system (may or may not be) in response to the input signal is called the output signal. For example, displacement due to force, output voltage from an amplifier, sinusoidal signal from an oscillator, etc.

2.1.5 Sampling

Sampling is a process by which a continuous-time signal (continuous with respect to time) is converted into a sequence of discrete samples, with each sample representing the amplitude of the signal at a particular instant of time. The sampling can be either uniform or non-uniform sampling.

In uniform sampling, the space between any two samples is fixed throughout the signal under consideration. A uniform sampling is illustrated in Fig. 2.4. In nonuniform sampling, the space between any two samples varies throughout the signal under consideration based on their characteristics like frequency, etc. In general, uniform sampling is preferred over nonuniform sampling since it is simple to analyze and easy to implement. The hardware complexity is also low in uniform sampling.

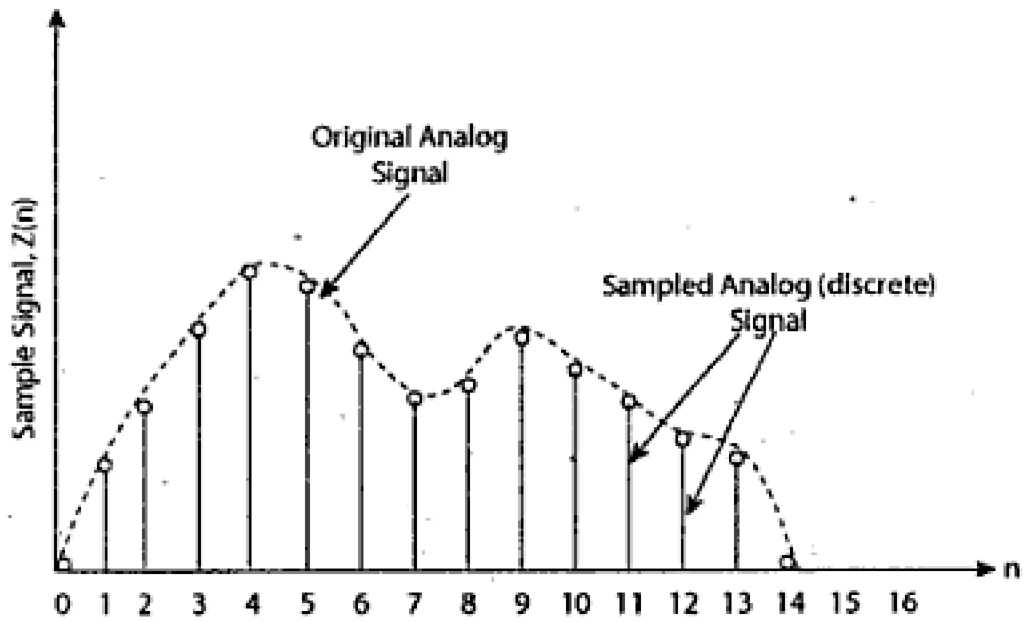


Fig. 2.4 Uniform Sampling of signal

2.1.6 Quantization

Quantization is a process by which each sample produced by the sampling circuit to the nearest level is selected from a finite number of discrete amplitude level as illustrated in Fig. 2.5.

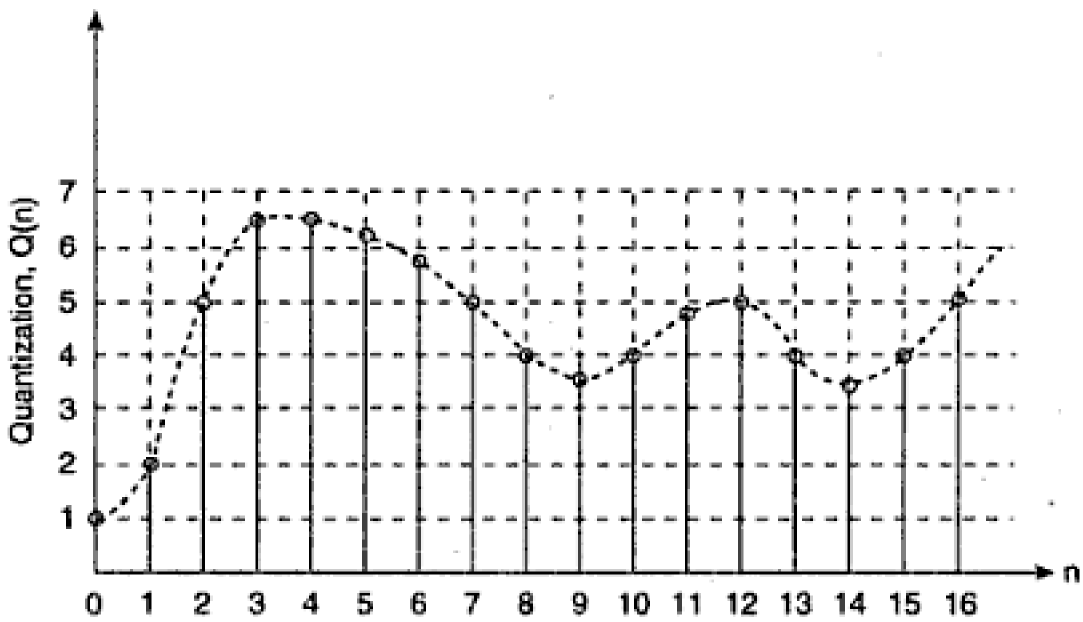


Fig. 2.5 Quantization of Signal

2.1.7 Coding

Coding is needed in order to represent each quantized sample by a binary number '0' or '1'. The '0' represents the "low" state or logical '0', and '1' represents the "high" state or logical '1'. The encoded version of quantized signal of Fig. 2.5 is shown in Table 2.1.

Table 2.1 3-bit Quantization and its Binary Representation

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Q(n)	1	2	5	7	7	7	6	5	4	4	4	5	5	4	4	4	5
Binary	001	010	101	111	111	111	110	101	100	100	100	101	101	100	100	100	101

2.2 CLASSIFICATION OF SIGNALS

Signals are classified based on their fundamental properties. They are:

1. Continuous-time signal and Discrete-time signal
2. Periodic signal and Aperiodic signal
3. Even signal and Odd signal
4. Deterministic signal and Random signal
5. Energy signal and Power signal

2.2.1 Continuous-time Signal and Discrete-time Signal

Signal can be represented either by continuous or discrete values.

Continuous-time signal A signal $x(t)$ is said to be a continuous-time signal if it is defined for all time t. The amplitude of the signal varies continuously with time. In general, all signals by nature are continuous-time signals.

The speech signal is a continuous-time signal, that is, conversation between persons is continuous with respect to time (Fig. 2.6a).

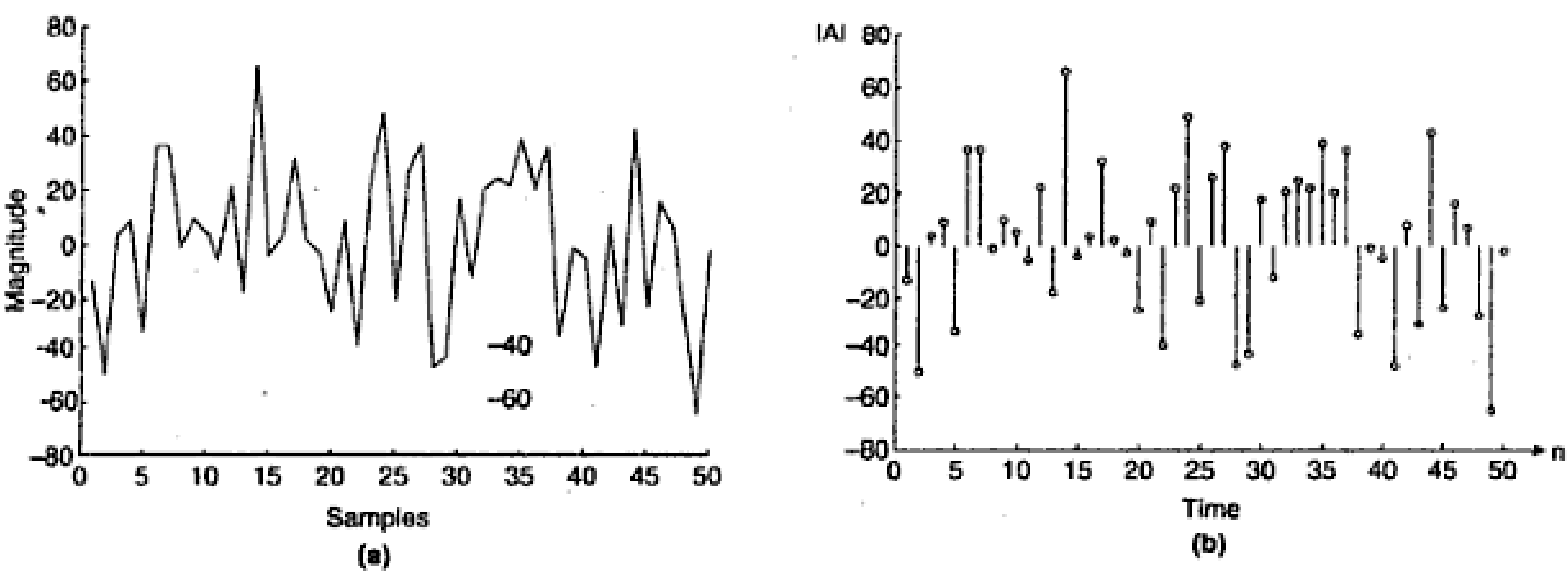


***Fig. 2.6** (a) Continuous-time Signal Representation of Speech Signal*
(b) Discrete-time Signal Representation of Speech Signal

The electrocardiogram, which is the electrical representation of the cardiac muscle, is continuous with respect to time (Fig. 2.7(a)).

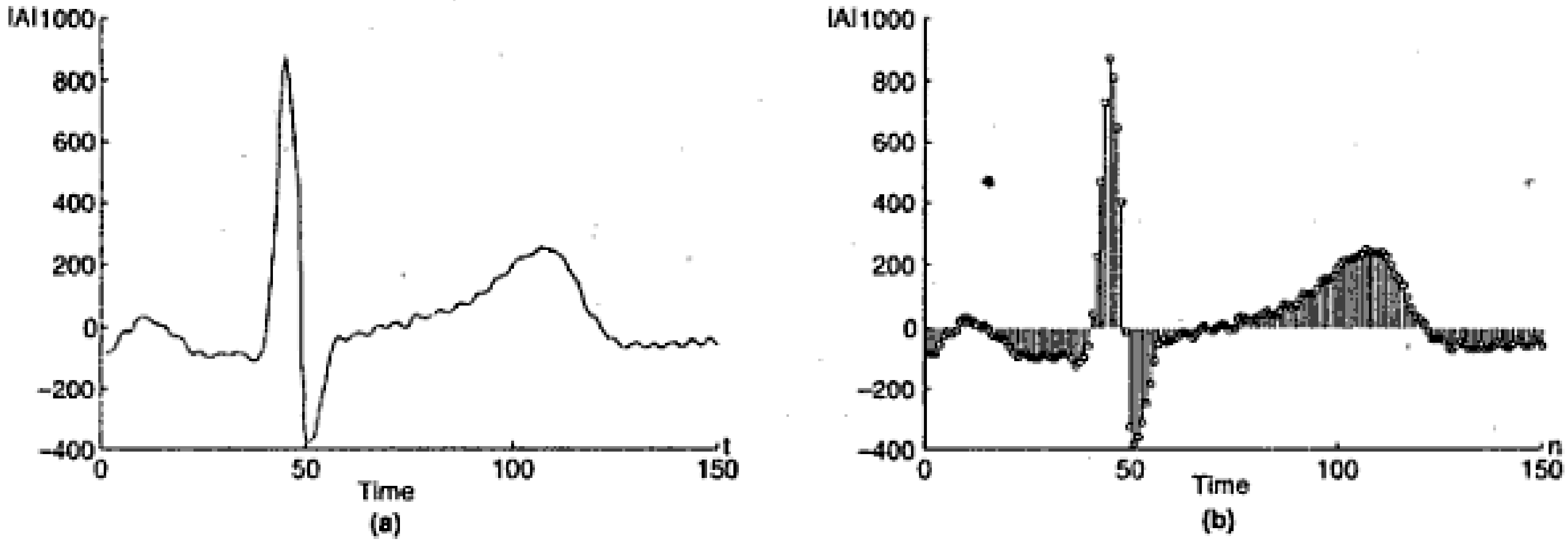


Fig. 2.7 (a) Continuous-time Signal Representation of Electrocardiogram
(b) Discrete-time Signal Representation of Electrocardiogram

The sinusoidal signal as illustrated in Fig. 2.8(a) is continuous with respect to time.

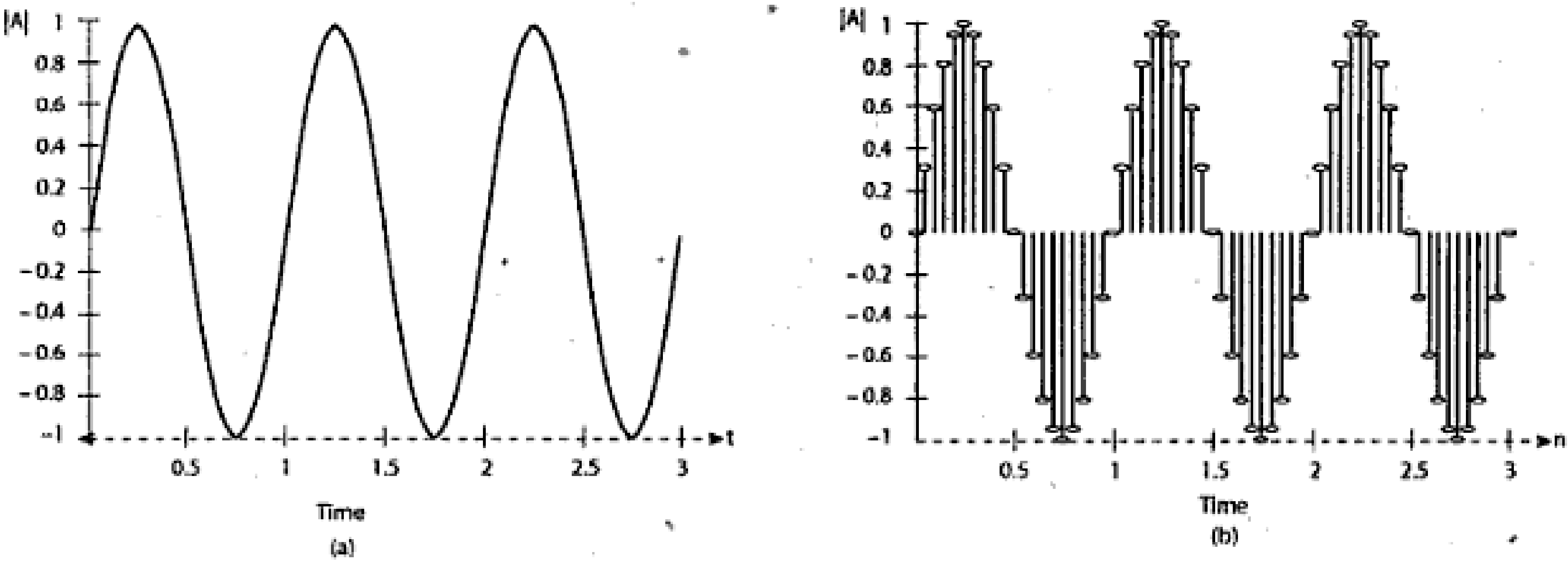


Fig. 2.8 (a) Continuous-time Signal Representation of Sinusoidal Signal
(b) Discrete-time Signal Representation of Sinusoidal Signal

Discrete-time signal Most of the signals that are obtained from their sources are continuous in time. Such signals have to be discretised since the processing done on the digital computer is digital in nature. A signal $x(n)$ is said to be discrete-time signal if it can be defined for a discrete instant of time (say n). For a discrete-time signal, the amplitude of the signal varies at every discrete value n, which is generally uniformly spaced. A discrete-time signal $x(n)$ is often obtained by sampling the continuous-time signal $x(t)$ at a uniform or nonuniform rate. The discrete-time representation of speech signal, electrocardiogram and sinusoidal signal is shown in Fig. 2.6(b), 2.7(b) and 2.8(b) respectively.

A continuous-time signal $x(t)$ can be converted to discrete-time signal $x(n)$ by substituting $t=nT$, that is,

$$x(t) = x(nT)\,|_{t=nT} = x(n) \qquad (2.1)$$

$$n = 0,\ \pm 1,\ \pm 2,\ +3,\ldots$$

where n = Constant integer, which can take positive or/and negative values

T = Sampling period, is an integer (normally T is assume to be unity)

SOLVED PROBLEMS

Problem 2.1 The continuous-time signal $x(t) = 5\sin(\pi t)$ for the interval $3 \geq t \geq 0$. Plot the corresponding discrete-time signal with a sampling period $T = 0.1$ s.

Solution

$$x(t) = 5\sin(\pi t) \text{ for } 3 \geq t \geq 0$$

t	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
$x(t)$	0	2.9	4.7	4.7	2.9	0	–2.9	–4.7	–4.7	–2.8	0	2.9	4.7	4.7	2.9	0

The plot of continuous-time signal can be obtained by connecting each point by a line, as shown in Fig. 2.9 (a).

The discrete-time signal can be obtained by a simple calculation given below:

$$x(t) = x(nT)\,|_{t=nT}$$

$$x(t) = x(0.1n)\,|_{t=0.1n}$$

$$x(n) = 5\sin(\pi nT) = 5\sin(0.1\pi n) \qquad n = 0,\ \pm 1,\ \pm 2,\ldots$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$x(n)$	0	1.5	2.9	4	4.8	5	4.8	4	2.9	1.5	0	–1.5	–2.9	–4	–4.8	–5

The plot of the discrete-time signal is shown in Fig. 2.9 (b).

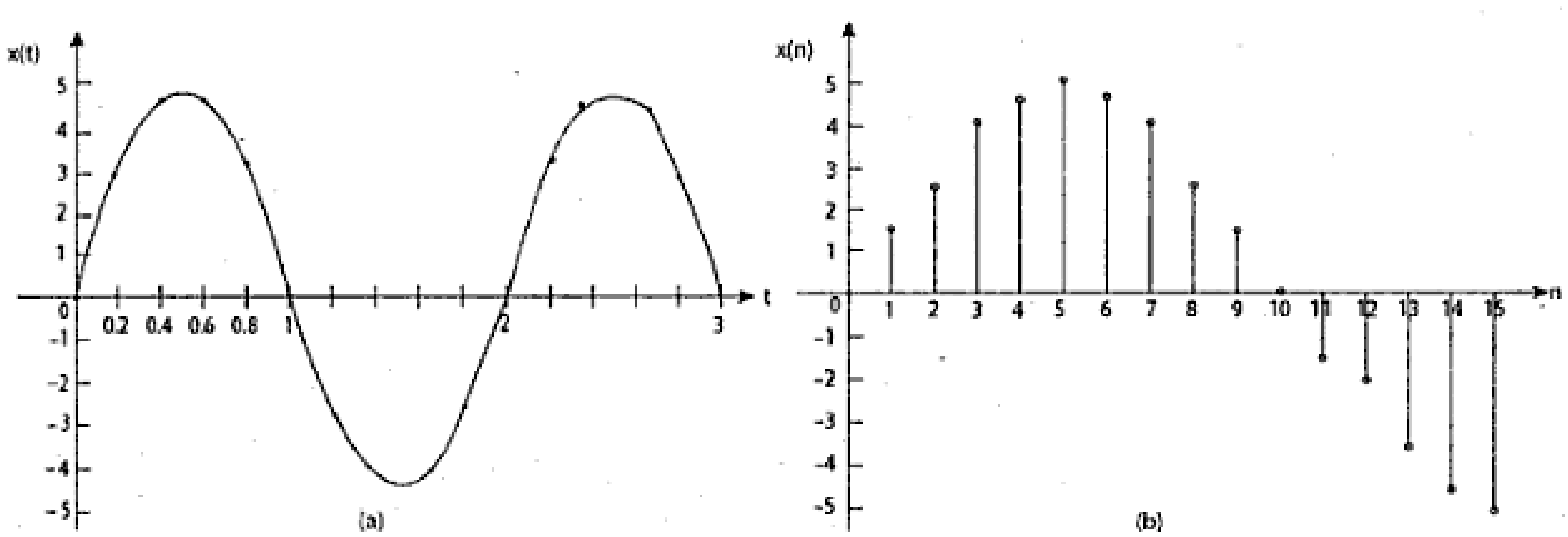


Fig. 2.9 (a) Continuous-time Signal (b) Discrete-time Signal

Problem 2.2 The continuous-time signal $x(t) = e^{-2t}$ for the interval $2 \geq t \geq -2$. Plot the corresponding discrete-time signal with a sampling period $T = 0.1$ s.

Solution

$$x(t) = e^{-2t} \text{ for } 2 \geq t \geq -2$$

t	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
$x(t)$	54.6	20	7.4	2.7	1	0.37	0.13	0.05	0.02

The plot of the above continuous-time signal can be obtained by connecting each point by a line, as shown in Fig. 2.10 (a).

The discrete-time signal can be obtained by

$$x(t) = x(nT)\,|_{t=nT}$$
$$x(t) = x(0.1n)\,|_{t=0.1n}$$
$$x(n) = e^{-2(0.1n)} = e^{-0.2n} \qquad n = 0, \pm 1, \pm 2, \ldots$$

n	−5	−4	−3	−2	−1	0	1	2	3	4	5
$x(t)$	2.7	2.23	1.82	1.5	1.2	1	0.8	0.67	0.5	0.45	0.37

The plot of the discrete-time signal is shown in Fig. 2.10 (b).

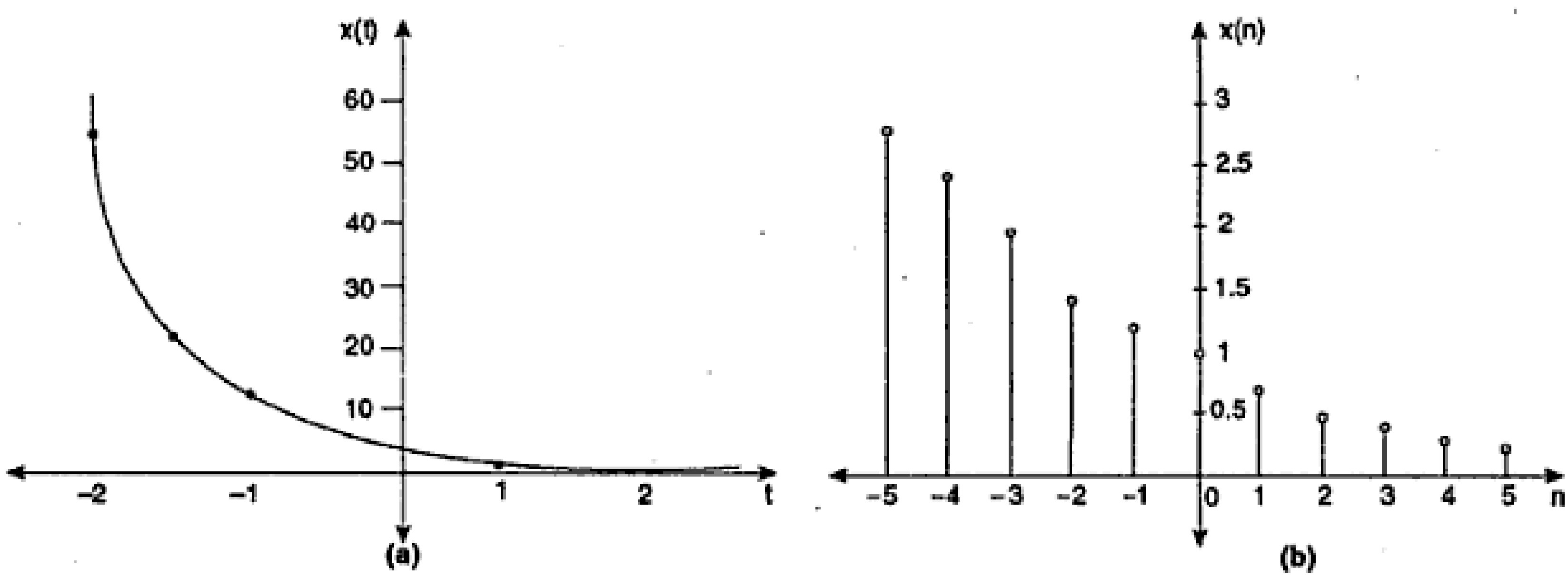


Fig. 2.10 (a) Continuous-time Signal (b) Discrete-time Signal

2.2.2 Periodic and Aperiodic Continuous-time Signal

A continuous-time signal $x(t)$ is said to be periodic if

$$x(t) = x(t + T),\ T > 0 \tag{2.2}$$

for all values of t,

where T = period of a cycle, which is an integer value

$$x(t) = x(t+T) = x(t+2T) = x(t+3T) = \ldots = x(t+nT) \tag{2.3}$$

where n = any integer

Hence, a periodic signal with period $T > 0$ is also periodic with period nT.

Prove that the cosine signal is periodic with periodicity *T*

Let us consider a cosine signal $x(t) = A\cos(\omega t + \phi)$

Let us assume period T which is same for all cycle in the given cosine signal

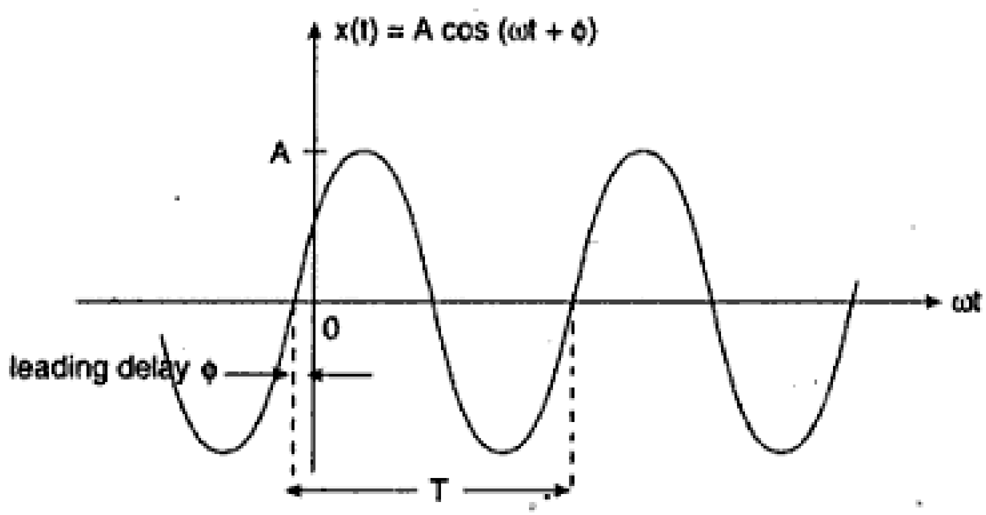


Fig. 2.11 A cosine signal

$$x(t+T) = A\cos\{\omega(t+T)+\phi\}$$
$$x(t+T) = A\cos\{\omega t+\omega T+\phi\}$$
$$x(t+T) = A\cos\{\omega t+2\pi+\phi\}$$
$$x(t+T) = A\cos\{\omega t+\phi\} = x(t)$$

The cosine signal is periodic as it satisfies periodicity equation (2.2).

SOLVED PROBLEM

Problem 2.3 Test whether the given signals are periodic or not.

(i) $x(t) = e^{\sin(t)}$ (ii) $x(t) = te^{\sin(t)}$

Solution

(i) $x(t) = e^{\sin(t)}$

From the definition of periodicity, $x(t) = x(t + T)$ for $T > 0$

Substitute $t = (t + T)$,

$$x(t+T) = e^{\sin(t+T)}$$

Since $T = 2\pi$, $\sin(t+T) = \sin(t+2\pi) = \sin(t)$

Therefore, $x(t+T) = e^{\sin(t+T)} = e^{\sin(t)} = x(t)$

Hence, the signal $x(t) = e^{\sin(t)}$ is periodic.

(ii) $x(t) = te^{\sin(t)}$

From the definition of periodicity, $x(t) = x(t+T)$ for $T > 0$

Substitute $t = (t + T)$,

$$x(t+T) = (t+T)\,e^{\sin(t+T)}$$

Since $T = 2\pi$, $\sin(t+T) = \sin(t+2\pi) = \sin(t)$

Therefore, $x(t+T) = (t+T)\,e^{\sin(t+T)} = (t+T)e^{\sin(t)} \neq x(t)$

Hence, the signal $x(t) = te^{\sin(t)}$ is aperiodic.

Discrete-time periodic signal A discrete-time signal is said to be periodic with period N, if it is unchanged by a time shift of N, i.e.

$$x(n) = x(n+N), \quad \text{for all } n \tag{2.4}$$

where N is a positive integer.

Fundamental period The fundamental period T_0 of the continuous-time signal $x(t)$ is the smallest positive value of T for which equation (2.2) holds. Any signal $x(t)$ for which there is no value of T to satisfy the condition of equation (2.2) is called an aperiodic signal.

The fundamental period N_0 of the discrete-time signal $x(n)$ is the smallest positive value of N for which equation (2.4) holds. Any signal $x(n)$ for which there is no value of N to satisfy the condition of equation (2.4) is called an aperiodic signal in discrete sense.

SOLVED PROBLEMS

Problem 2.4 Test whether the given exponential is periodic or not.

$$x(t) = e^{j\omega_0 t}$$

Solution By definition, $x(t)$ will be periodic if, $e^{j\omega_0(t+T)} = e^{j\omega_0 t}$ (1)

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}$$

In order to satisfy equation (2.2), $e^{j\omega_0 T} = 1$

We know that, $e^{j\omega_0 T} = \cos\omega_0 T + j\sin\omega_0 T$

For $\omega_0 = 0$, $e^{j\omega_0 T} = 1$ (ω_0=0 defines only DC signal)

For $\omega_0 \neq 0$, $\omega_0 T = 2\pi m$ (defines AC signal)

then $e^{j\omega_0 T} = 1$

where $m = 1, 2, 3, \ldots$ (integers)

Therefore, $$e^{j\omega_0(t+T)} = e^{j\omega_0 t}\Big|_{\omega_0 T = 2\pi m}$$

$$\omega_0 T = 2\pi m$$

Therefore, periodicity is given by, $$\boxed{T = \frac{2\pi}{\omega_0} m}$$ (2)

Similarly, for discrete-time signal $x(n)$, the condition for periodicity is given by

$$\Omega_o N = 2\pi m$$

$$N = \frac{2\pi}{\Omega_o} m$$

where N is an integer always.

Problem 2.5 Test whether the signal is periodic. If so, find the fundamental period.

$$x(t) = \cos\left(t + \frac{\pi}{3}\right)$$

Solution The given signal resembles the general expression, $x(t) = \cos(\omega t + \phi)$.

Therefore, on comparing the given problem with general expression the frequency $\omega = 1$.

The fundamental period for which the given signal exhibits periodicity is,

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{1} = 2\pi$$

The given signal is periodic with periodicity 2π.

Problem 2.6 Test whether the signal is periodic. If so, find the fundamental period.

$$x(t) = \sin\left(\frac{2\pi}{5}\right)t$$

Solution

$$x(t) = \sin\left(\frac{2\pi}{5}\right)t$$
$$\uparrow\ \omega_0$$

The frequency, $\omega_o = \dfrac{2\pi}{5}$

Therefore, $$T_0 = \frac{2\pi}{\omega_0} = \left(\frac{2\pi}{2\pi/5}\right) = 5$$

The fundamental period for which the given signal exhibits periodicity is $T_0 = 5$.

Problem 2.7 Find the fundamental period of $x(t) = \cos\left(\dfrac{\pi}{3}\right)t + \sin\left(\dfrac{\pi}{5}\right)t$

$$x(t) = \underbrace{\cos\left(\frac{\pi}{3}\right)t}_{x_1(t)} + \underbrace{\sin\left(\frac{\pi}{5}\right)t}_{x_2(t)}$$

Solution The given signal $x(t)$ is a composite signal which has two component signals $x_1(t)$ and $x_2(t)$.

For $x_1(t)$, the frequency, $\omega_1 = \dfrac{\pi}{3}$

The fundamental period of $x_1(t)$ is $T_1 = \dfrac{2\pi}{\omega_1} = \dfrac{2\pi}{(\pi/3)} = 6$

For $x_2(t)$, the frequency, $\omega_2 = \dfrac{\pi}{5}$

The fundamental period of $x_2(t)$ is $T_2 = \dfrac{2\pi}{\omega_2} = \dfrac{2\pi}{(\pi/5)} = 10$

The signal $x(t)$ is said to be periodic if and only if the ratio of T_1 to T_2 is a rational, that is,

$$\frac{T_1}{T_2} = \frac{6}{10} = \frac{3}{5}\ \text{(rational)}$$

The fundamental period of given signal $x(t)$ is, $T_0 = 5T_1 = 3T_2 = 30$.

Problem 2.8 Test whether the given signal is periodic or not.

$$x(t) = \cos t + \sin\sqrt{2}\,t$$

$$x(t) = \underbrace{\cos t}_{x_1(t)} + \underbrace{\sin\sqrt{2}t}_{x_2(t)}$$

Solution The given signal $x(t)$ is a composite signal which has two component signals $x_1(t)$ and $x_2(t)$.

For $x_1(t)$, the frequency, $\omega_1 = 1$

The fundamental period of $x_1(t)$ is $T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{1} = 2\pi$

For $x_2(t)$, the frequency, $\omega_2 = \sqrt{2}$

The fundamental period of $x_2(t)$ is $T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

The signal $x(t)$ is said to be periodic if and only of the ratio of T_1 to T_2 is a rational, that is,

$$\frac{T_1}{T_2} = \frac{2\pi}{\sqrt{2}\,\pi} = \frac{\sqrt{2}}{1} \text{ (irrational)}$$

Since the ratio of T_1 to T_2 exhibits an irrational ratio, therefore signal $x(t)$ cannot be periodic.

Note *Any signal exhibits irrational ratio of their period is aperiodic signal.*

Problem 2.9 Test for periodicity of $x(t) = je^{j10t}$ (ω_0)

Solution The frequency of the signal, $\omega_o = 10$

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{10} = \frac{\pi}{5}$$

The signal is periodic with fundamental period $\pi/5$.

Problem 2.10 Test whether the given discrete-signal is periodic. If so, find the fundamental period.

$$x(n) = \sin\left(\frac{2\pi}{3}\right)n$$

Solution

For the given signal, the frequency, $\Omega_0 = \frac{2\pi}{3}$

where Ω_0 is the frequency in radians in the discrete domain.

The fundamental period, $N_0 = \frac{2\pi m}{\Omega_0} = \frac{2\pi m}{2\pi/3} = 3 \quad (m = 1)$

Note *The fundamental period N of any discrete signal must be an integer.*

Problem 2.11 Test for the periodicity of $x(n) = \cos^2\left(\frac{\pi}{8}\right)n$

Solution

$$x(n) = \cos^2\left(\frac{\pi}{8}\right)n$$

$$x(n) = \frac{1 + \cos 2\left(\frac{\pi}{8}\right)n}{2}$$

Hint $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$$x(n) = \frac{1}{2}\left[1 + \cos\left(\frac{\pi}{4}\right)n\right]$$

Since the frequency, $\Omega_0 = \frac{\pi}{4}$,

$$N_0 = \frac{2\pi m}{\Omega_0} = \frac{2\pi m}{\pi/4} = 8 \qquad (m = 1)$$

The given signal is periodic with fundamental period $N_0 = 8$.

Problem 2.12 Test for the periodicity of $x(n) = e^{j7\pi n}$

Solution

For the given signal, the frequency, $\Omega_0 = 7\pi$

$$N_0 = \frac{2\pi m}{\Omega_0} = \frac{2\pi m}{7\pi} = \frac{2}{7}(m = 1)$$

Note *The general expression for fundamental period is* $N_0 = \frac{2\pi}{\Omega_0} m$. *In the previous problems, the value of m was assumed to be 1. In this problem, in order to satisfy the integer condition for a fundamental period, m is assumed to be 7.* That is,

$$N_0 = \frac{2\pi}{7\pi} \times \underset{\uparrow m}{7} = 2$$

Therefore, the fundamental period $N_0 = 2$ if $m = 7$

Problem 2.13 Test for the periodicity of $x(n) = 4e^{j\frac{4\pi\left(n+\frac{1}{3}\right)}{5}}$

Solution

$$x(n) = 4e^{j\frac{4\pi\left(n+\frac{1}{3}\right)}{5}}$$

$$x(n) = 4e^{j\frac{4\pi}{5}n} \cdot e^{j\frac{4\pi}{15}}$$

Phase

Frequency

Note *Phase term never contributes in the definition of periodicity, that is, irrespective of the phase term the given signal may or may not be periodic.*

For the given signal, the frequency, $\Omega_0 = \frac{4\pi}{5}$

Therefore, $$N_0 = \frac{2\pi m}{\Omega_0} = \frac{2\pi m}{\frac{4\pi}{5}} = \frac{5m}{2}$$

Period N_0 must be an integer. Therefore $m = 2$.

The fundamental period, $N_0 = \frac{5}{2} \times \underset{m}{2} = 5$

The given signal is periodic with period 5 if $m = 2$

Problem 2.14 Test for the periodicity of $x(n) = 4e^{j\frac{4\left(n+\frac{1}{3}\right)}{5}}$

Solution

$$x(n) = 4e^{j\frac{4}{5}n} \cdot e^{j\frac{4}{15}}$$

Phase

Frequency

For the given signal, the frequency, $\Omega_0 = \frac{4}{5}$

Therefore, $$N_0 = \frac{2\pi}{\Omega_0} = \frac{2\pi}{4/5} = \frac{5\pi}{2}$$

For m=7 (Substitute $\pi = \frac{22}{7}$), $N_0 = 55$

Hence, the given signal is periodic.

Problem 2.15 Determine the fundamental period of the signal

$$x(n) = 1 + e^{j\frac{4\pi n}{7}} - e^{j\frac{2\pi n}{5}}$$

Solution

$$x(n) = 1 + \underbrace{e^{j\frac{4\pi}{7}n}}_{x_1(t)} - \underbrace{e^{j\frac{2\pi}{5}n}}_{x_2(t)}$$

DC Component

The given signal $x(t)$ is a composite signal, contains three signals: DC component, signals $x_1(t)$ and $x_2(t)$. The periodicity is calculated based only on signals $x_1(t)$ and $x_2(t)$.

For the signal $x_1(n)$, the frequency $\Omega_1 = \frac{4\pi}{7}$

The periodicity of $x_1(n)$ is $$N_1 = \frac{2\pi}{\Omega_1}m = \frac{2\pi}{4\pi/7}m = \frac{7}{2}m$$

For $m = 2$, $N_1 = 7$

For the signal the $x_2(n)$, the frequency $\Omega_2 = \frac{2\pi}{5}$

The periodicity of $x_2(n)$ is $N_2 = \frac{2\pi}{\Omega_0} m = \frac{2\pi}{2\pi/5} = 5 \quad (m = 1)$

The signal is said to be periodic if and only if the ratio of N_1 to N_2 is a rational, that is,

$$\frac{N_1}{N_2} = \frac{7}{5} \text{ (rational)}$$

The fundamental period is $N_0 = 5N_1 = 7N_2 = 35$

Problem 2.16 A pair of sinusoidal signals with a common angular frequency is represented by

$$x_1(n) = 3\sin(3\pi n) \text{ and } x_2(n) = \sqrt{3}\,\sin(3\pi n)$$

Specify the period for which the period N of both $x_1(n)$ and $x_2(n)$ must satisfy them to be periodic.

Solution The common angular frequency of both the signal is given by

$$\omega_0 = 3\pi \text{ rad/s}$$

Phase angle, $\phi = 0$

We know that the period N for discrete-time signal is given by

$$N = \left(\frac{2\pi}{\Omega_0}\right) m = \left(\frac{2\pi}{3\pi}\right) m = \left(\frac{2}{3}\right) m$$

For $x_1(n)$ and $x_2(n)$ to be periodic, their period N must be an integer. This can only be satisfied for $m = 3$, 6, 9, 12, 15, ... (integer multiples of 3), which results in period $N = 2, 4, 6, \ldots$ (integer multiples of 2).

Problem 2.17 Determine whether the signal $x(n) = \cos(0.1\pi n)$ is periodic or not.

Solution

$$x(n) = \cos(\underbrace{0.1\pi}_{\Omega_0} n)$$

For the signal $x(n)$, the frequency $\Omega_0 = 0.1\pi$

The periodicity of $x(n)$ is $N = \frac{2\pi}{\Omega_0} = \frac{2\pi}{0.1\pi} = 20$

The given signal is periodic with period 20.

2.2.3 Even and Odd Continuous-time Signal

A continuous-time signal $x(t)$ is said to be even, if it satisfies the condition

$$x(t) = x(-t), \text{ for all } t \tag{2.5}$$

Even signals are symmetric about the vertical axis, as illustrated in Fig. 2.12 (a). A continuous-time signal $x(t)$ is said to be odd, if it satisfies the condition

$$x(t) = -x(-t), \text{ for all } t \tag{2.6}$$

Odd signals are antisymmetric about the vertical axis, as illustrated in Fig. 2.12 (b).

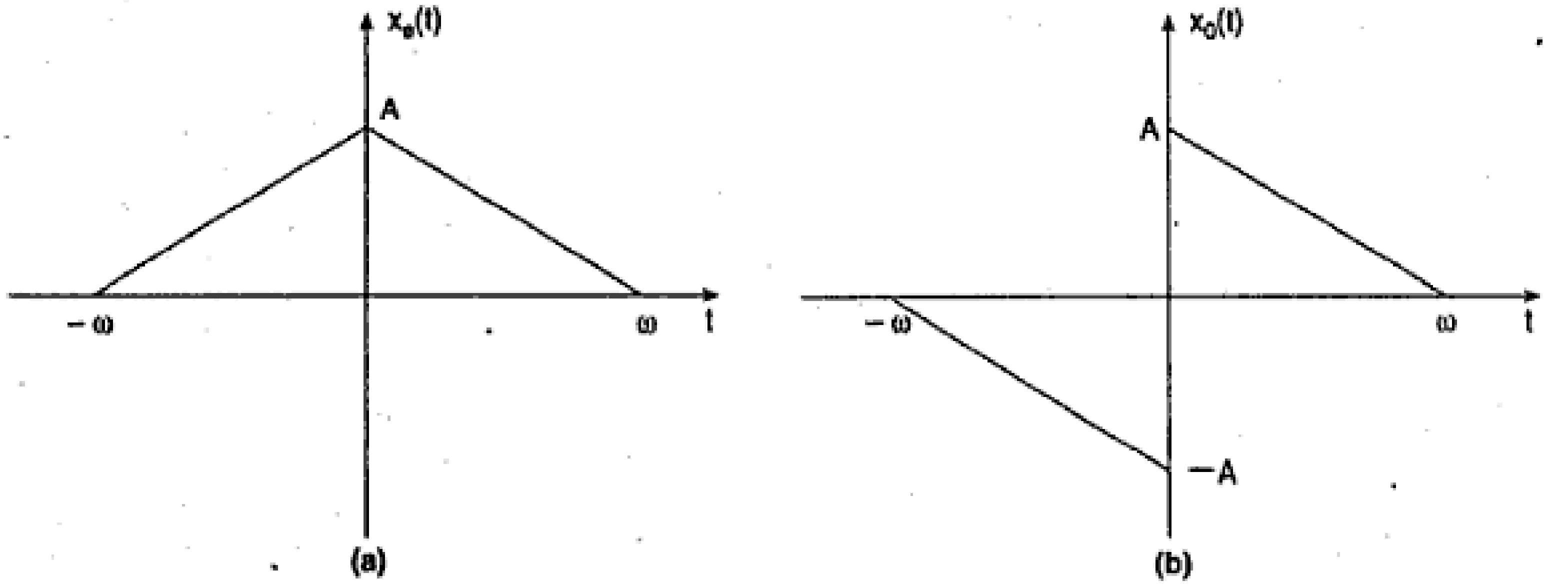


***Fig. 2.12** (a) Even Signal (b) Odd Signal*

Let us consider a signal, $x(t)$ which can be decomposed as odd and even signals, i.e.

$$x(t) = x_o(t) + x_e(t) \tag{2.7}$$

where $x_o(t)$ represents the odd signal and $x_e(t)$ represents the even signal.

From the definition of odd and even signal,

$$-x_o(t) = x_o(-t) \tag{2.8a}$$

$$x_e(t) = x_e(-t) \tag{2.8b}$$

Replace $t = -t$ in equation (2.7),

$$x(-t) = x_o(-t) + x_e(-t) \tag{2.9}$$

Substituting equation (2.8) in (2.9),

$$x(-t) = -x_o(t) + x_e(t) \tag{2.10}$$

Solve equations (2.7) and (2.10).

Adding (2.7) and (2.10), $\quad x(t) + x(-t) = 2\,x_e(t)$

Subtracting (2.7) from (2.10),

$$x(t) - x(-t) = 2\,x_o(t)$$

Solve for $x_e(t)$ and $x_o(t)$,

$$x_e(t) = \frac{x(t) + x(-t)}{2} \tag{2.11}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} \tag{2.12}$$

Equations (2.11) and (2.12) gives the relation between odd and even signals.

Complex valued continuous-time signal A complex valued signal $x(t)$ is said to be conjugate symmetry if it satisfies the condition

$$x(-t) = x^*(t)$$

where $x^*(t)$ is the complex conjugate of $x(t)$.

If $$x(t) = x_R(t) + jx_I(t) \tag{2.13}$$

where $x_R(t)$ is real part of $x(t)$ and $x_I(t)$ is imaginary part of $x(t)$

On conjugating equation (2.13),

$$x^*(t) = x_R(t) - jx_I(t) \tag{2.14}$$

It is shown that conjugation property affects only the imaginary part $x_I(t)$ of the signal $x(t)$, not the real part $x_R(t)$ of the signal. It is also understood that a complex valued signal $x(t)$ is conjugate symmetric if its real part is an even signal and its imaginary part is an odd signal.

Odd and even discrete-time signal A discrete-time signal $x(n)$ is said to be even, if it satisfies the condition

$$x(n) = x(-n)\text{, for all } n$$

Even signals are symmetric about the vertical axis as illustrated in Fig. 2.13 (a).

A discrete-time signal $x(n)$ is said to be odd, if it satisfies the condition

$$x(n) = -x(-n)\text{, for all } n$$

Odd signals are antisymmetric about the vertical axis as illustrated in Fig. 2.13 (b).

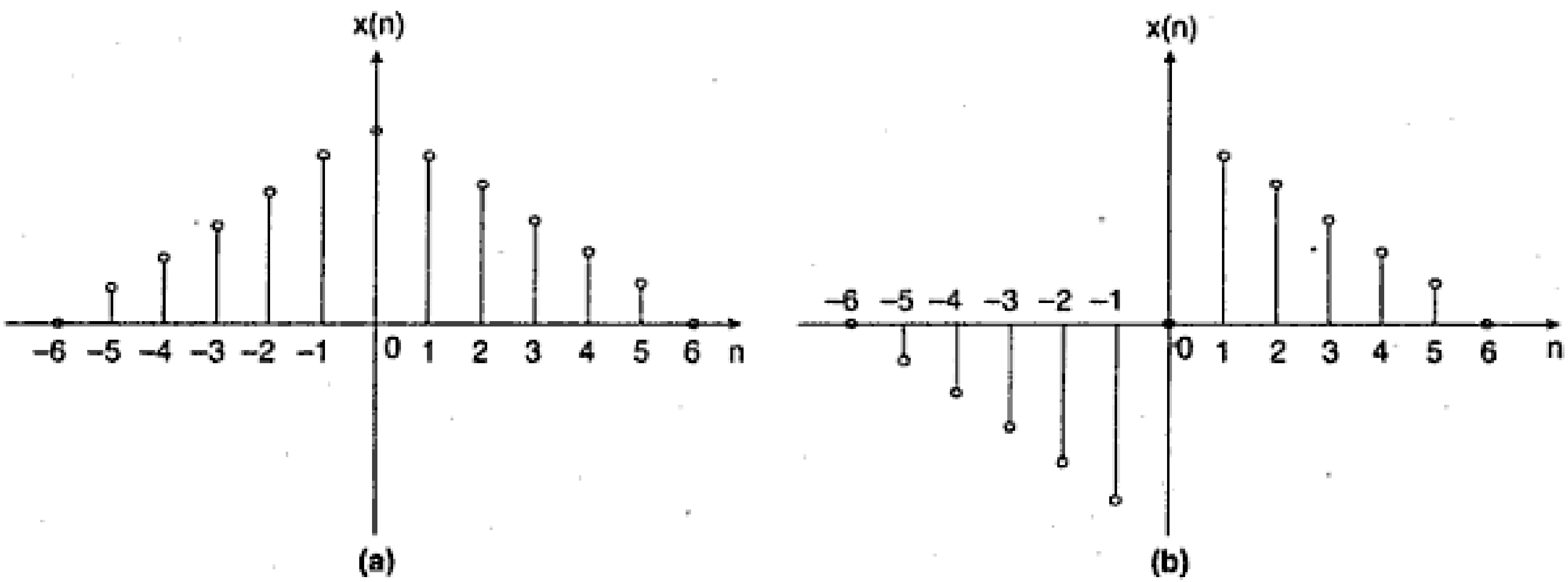


Fig. 2.13 ***(a) Even Signal (b) Odd Signal***

Let us consider a signal $x(n)$, which can be decomposed as odd and even signals, i.e.

$$x(n) = x_o(n) + x_e(n) \tag{2.15}$$

where $x_o(n)$ is odd signal and $x_e(n)$ is even signal

From the definitions of odd and even signal,

$$-x_o(n) = x_o(-n) \tag{2.16}$$

$$x_e(n) = x_e(-n) \tag{2.17}$$

Replace $n = -n$ in equation (2.15),

$$\begin{aligned} x(-n) &= x_o(-n) + x_e(-n) \\ x(-n) &= -x_o(n) + x_e(n) \end{aligned} \tag{2.18}$$

Adding equations (2.15) and (2.18),

$$x(n)+x(-n)=2x_e(n)$$

Subtracting equation (2.15) from (2.18),

$$x(n)-x(-n)=2x_o(n)$$

$$x_e(n)=\frac{x(n)+x(-n)}{2} \tag{2.19}$$

$$x_o(n)=\frac{x(n)-x(-n)}{2} \tag{2.20}$$

Equations (2.19) and (2.20) gives the relation between odd and even conditions.

SOLVED PROBLEMS

Problem 2.18 Find the odd and even components of $x(t)=e^{j2t}$

Solution We know that, any signal comprises of even and odd parts, i.e.

$$x(t)=x_e(t)+x_o(t)=e^{j2t}$$

The even signal is given by

$$x_e(t)=\frac{x(t)+x(-t)}{2}=\frac{e^{j2t}+e^{-j2t}}{2}=\cos 2t$$

The odd signal is given by

$$x_o(t)=\frac{x(t)-x(-t)}{2}=\frac{e^{j2t}-e^{-j2t}}{2}$$

$$x_o(t)=j\left[\frac{e^{+j2t}-e^{-j2t}}{2j}\right]=j\sin 2t$$

$$x(t)=x_e(t)+x_o(t)=\cos 2t+j\sin 2t$$

Problem 2.19 Show that the product of two even signals or two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal.

Solution Let us consider two even signals $x_1(t)$ and $x_2(t)$.

$$x(t)=x_1(t)\times x_2(t)$$

Replace $t=-t$, then

Hint For even signal $x(t)=x(-t)$

$$x(-t)=x_1(-t)\times x_2(-t)=x(t)$$

Hence, it is an even signal.

Let us consider two odd signals $x_3(t)$ and $x_4(t)$.

$$x(t)=x_3(t)\times x_4(t)$$

Replace $t = -t$, then

$$x(-t) = x_3(-t) \times x_4(-t)$$
$$x(-t) = [-x_3(t)] \times [-x_4(t)]$$
$$x(-t) = x_3(t) \times x_4(t) = x(t)$$

Hint For odd signal $x(-t) = -x(t)$

Hence, it is an even signal.

Let us consider two signals such that $x_1(t)$ is even and $x_3(t)$ is odd. Then

$$x(t) = x_1(t) \times x_3(t)$$

Replace $t = -t$, then

$$x(-t) = x_1(-t) \times x_3(-t)$$

$x(-t) = x_1(t) \times [-x_3(t)] = -x(t)$ which is odd.

Summary Even × Even = Even; Odd × Odd = Even; Even × Odd = Odd

Problem 2.20 Find the even and odd components of $x(t) = \cos t + \sin t$.

Solution

$$x(t) = \cos t + \sin t$$
$$x(-t) = \cos(-t) + \sin(-t) = \cos t - \sin t$$

The even part is given by

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \frac{(\cos t + \sin t) + (\cos t - \sin t)}{2} = \cos t$$

The odd part is given by

$$x_o(t) = \frac{x(t) - x(-t)}{2} = \frac{(\cos t + \sin t) - (\cos t - \sin t)}{2} = \sin t$$
$$x(t) = x_e(t) + x_o(t) = \cos t + \sin t$$

Problem 2.21 Find the even and odd components of $x(n) = \{3, 2, \underset{\uparrow}{1}, 4, 5\}$.

Note The arrow mark always shows the value for 0th position i.e.

Position	−2	−1	0	1	2
$x(n)$	3	2	1	4	5

Solution The even part is given by

$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

For $n=-2$, $$x_e(-2)=\frac{x(-2)+x(2)}{2}=\frac{3+5}{2}=4$$

For $n=-1$, $$x_e(-1)=\frac{x(-1)+x(1)}{2}=\frac{2+4}{2}=3$$

For $n=0$, $$x_e(0)=\frac{x(0)+x(0)}{2}=\frac{1+1}{2}=1$$

For $n=1$, $$x_e(1)=\frac{x(1)+x(-2)}{2}=\frac{4+2}{2}=3$$

For $n=2$, $$x_e(2)=\frac{x(2)+x(-2)}{2}=\frac{5+3}{2}=4$$

$$x_e(n)=\{4,3,1,3,4\}$$

The odd part is given by

$$x_o(n)=\frac{x(n)-x(-n)}{2}$$

For $n=-2$, $$x_e(-2)=\frac{x(-2)-x(2)}{2}=\frac{3-5}{2}=-1$$

For $n=-1$, $$x_e(-1)=\frac{x(-1)-x(1)}{2}=\frac{2-4}{2}=-1$$

For $n=0$, $$x_o(0)=\frac{x(0)-x(0)}{2}=0$$

For $n=1$, $$x_o(1)=\frac{x(1)-x(-1)}{2}=\frac{4-2}{2}=1$$

For $n=2$, $$x_o(2)=\frac{x(2)-x(-2)}{2}=\frac{5-3}{2}=1$$

$$x_o(n)=\{-1,-1,0,1,1\}$$

Adding $x_e(n)$ and $x_o(n)$ position-wise, we obtain the original signal $x(n)$.

Problem 2.22 Draw the odd and even representations for the given signal.

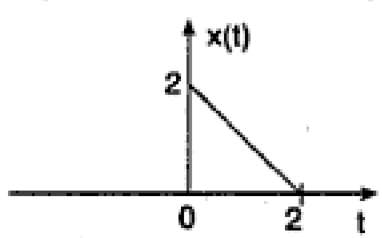


Fig. 2.14

Solution

For even signal,

$$x_e(t) = \frac{x(t)+x(-t)}{2}$$

For odd signal,

$$x_o(t) = \frac{x(t)-x(-t)}{2}$$

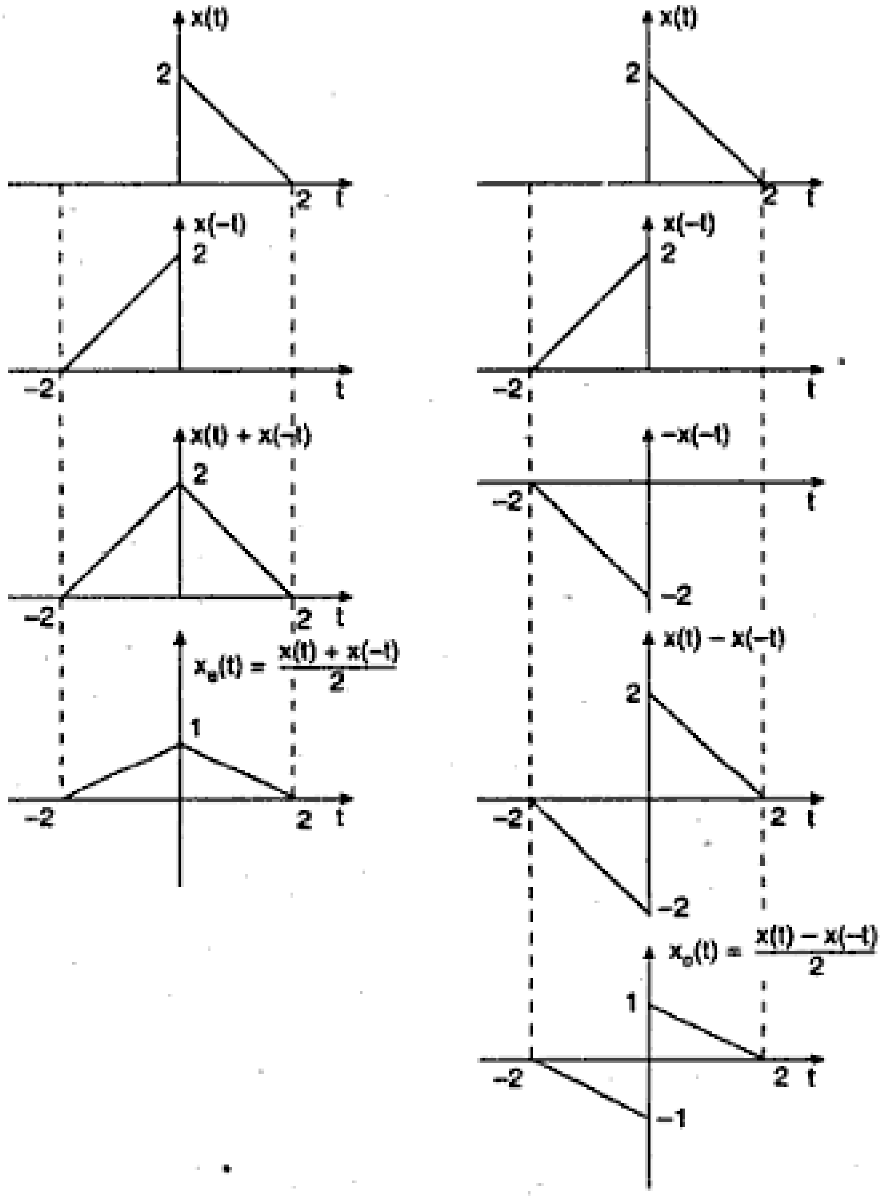


Fig. 2.15

Problem 2.23 Draw the even and odd signals of the signal shown below.

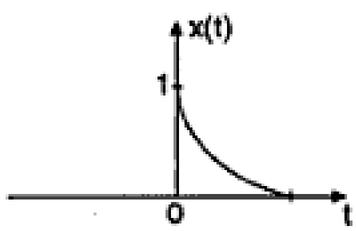


Fig. 2.16

Solution

For even signal,

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

For odd signal,

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

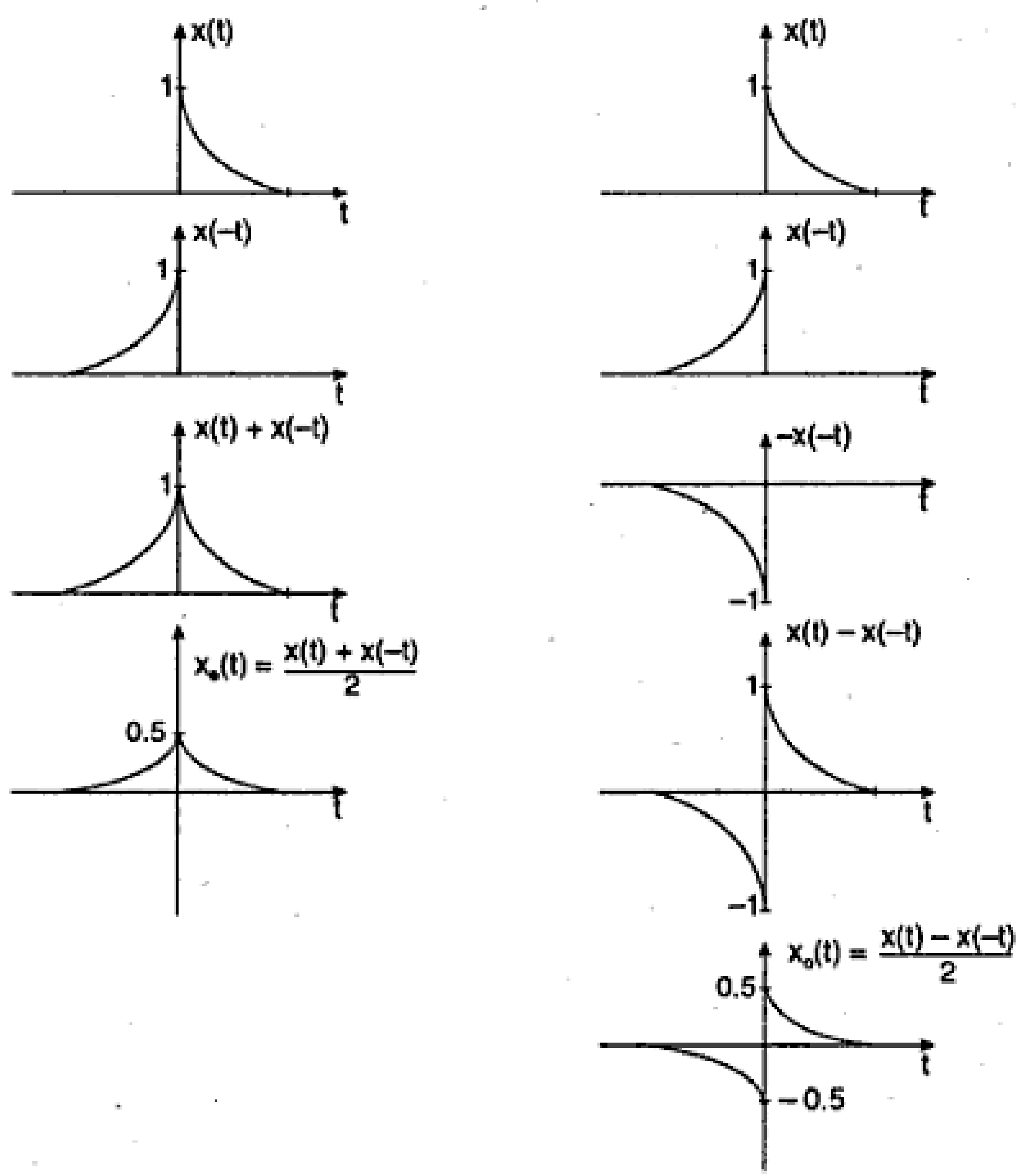


Fig. 2.17

Problem 2.24 Draw the even and odd signals of the given signal.

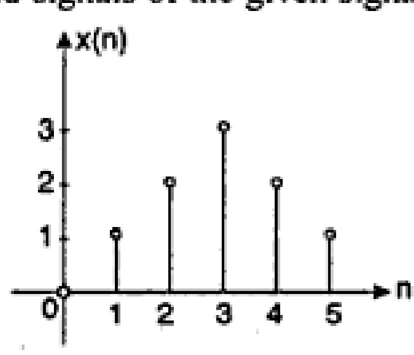


Fig. 2.18

Solution

For even signal,

$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

For odd signal,

$$x_o(n) = \frac{x(n) - x(-n)}{2}$$

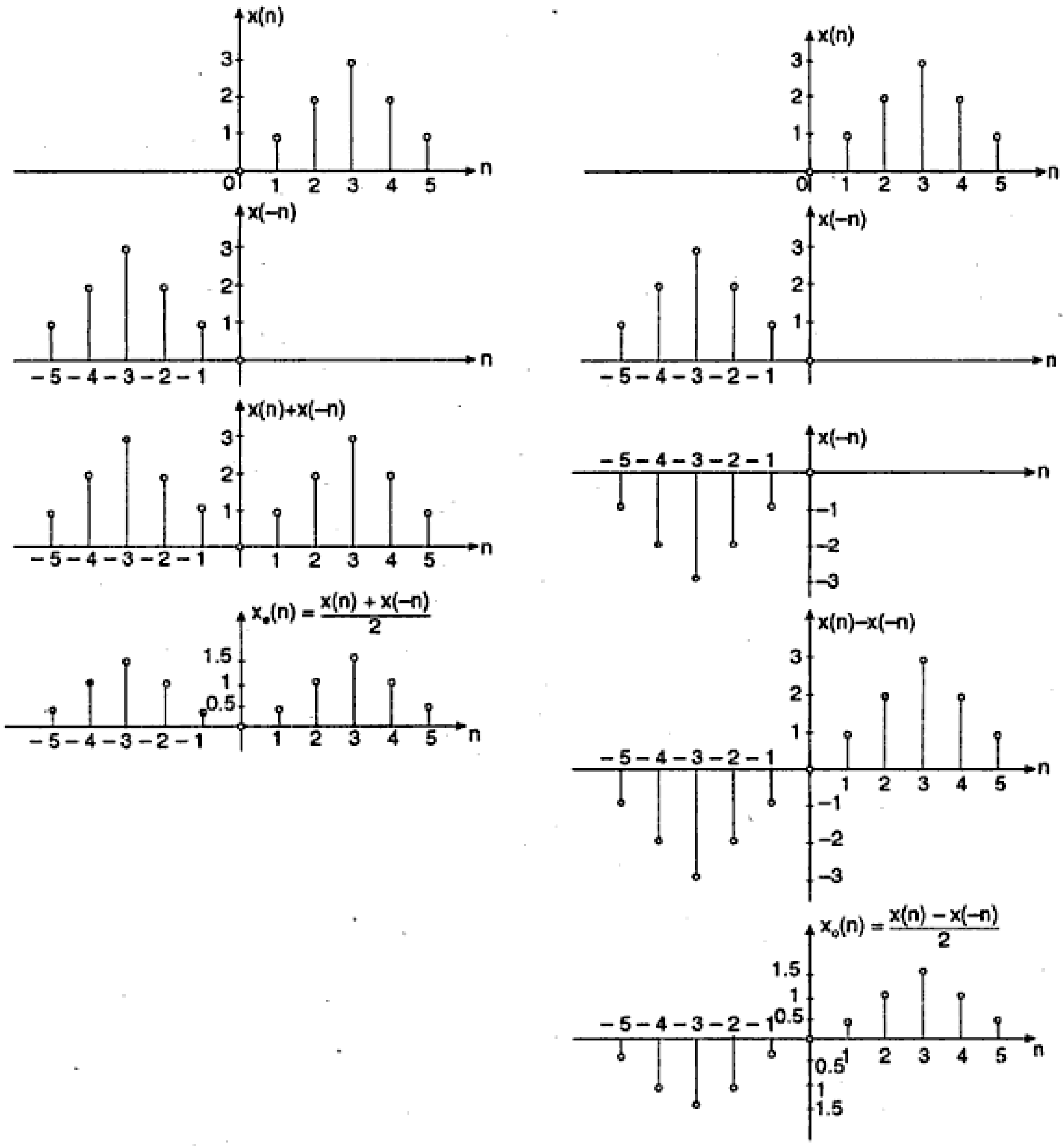


Fig. 2.19

Complex valued discrete-time signal A complex valued signal $x(n)$ is said to be conjugate symmetry if it satisfies the condition

$$x(-n) = x^*(n)$$

where, $x^*(n)$ is the complex conjugate of $x(n)$

If
$$x(n) = x_R(n) + jx_I(n) \tag{2.21}$$

where, $x_R(n)$ is real part of $x(n)$ and $x_I(n)$ is imaginary part of $x(n)$

On conjugating equation (2.21),

$$x^*(t) = x_R(n) - jx_I(n)$$

It is shown that the conjugation property affects only the imaginary part $x_I(n)$ of the signal $x(n)$, not the real part of the signal $x_R(n)$. It is also understood that a complex valued signal $x(n)$, is conjugate symmetric if its real part is an even signal and imaginary part is an odd signal.

2.2.4 Energy Signal and Power Signal

Let us consider a current $i(t)$ flowing through a resistor R, developing a voltage $v(t)$ across it.

The instantaneous power $p(t)$ dissipated in the resistor is given by

$$p(t) = \frac{v^2(t)}{R} = i^2(t)\ R \tag{2.22}$$

In signal processing or analysis, it is customary to define power dissipation in a resistor of unit value, i.e. $R = 1\,\Omega$, regardless of whether the given signal $x(t)$ represents a voltage $v(t)$ or a current $i(t)$. Hence, we can represent the instantaneous power $p(t)$ of the signal $x(t)$ as

$$p(t) = v^2(t) = i^2(t) = x^2(t) \tag{2.23}$$

The total energy of the continuous-time signal $x(t)$ expended over time interval $-T/2 \leq t \leq +T/2$ is given by

$$E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{+T/2} p(t)\,dt$$

$$E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{+T/2} |x(t)|^2\,dt$$

$$E_\infty = \int_{-\infty}^{+\infty} |x(t)|^2\,dt \tag{2.24}$$

For discrete-time signal $x(n)$, the total energy expended over time interval $-N \leq n \leq +N$ is given by

$$E = \underset{N\to\infty}{\mathrm{Lt}} \sum_{n=-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)^2 \tag{2.25}$$

The average power of the continuous-time signal $x(t)$, expended over the time interval $-T/2 \le t \le +T/2$ is given by

$$P=\underset{T\to\infty}{\mathrm{Lt}}\frac{1}{T}\int_{-T/2}^{+T/2} p(t)\,dt=\underset{T\to\infty}{\mathrm{Lt}}\frac{1}{T}\int_{-T/2}^{+T/2}|x(t)|^2\,dt$$

$$P_\infty=\frac{1}{T}\int_{-\infty}^{+\infty}|x(t)|^2\,dt$$

For discrete-time signal $x(n)$, the average power expended over the time interval $-N\le n\le +N$ is given by

$$P=\underset{N\to\infty}{\mathrm{Lt}}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2$$

A signal is referred to as a power signal if, and only if, the average power of the signal satisfies the condition

$$0<P<\infty$$

The square root of the average power P is called the root mean square (rms) value of the signal. It is also noted that periodic signals and random signals are usually viewed as a power signal, whereas deterministic signals and nonperiodic signals are energy signals.

SOLVED PROBLEMS

Problem 2.25 Test whether the given signal is an energy signal or a power signal.

$$x(t)=e^{-2t}u(t)$$

Solution

$$E=\underset{T\to\infty}{\mathrm{Lt}}\int_{-T/2}^{T/2}|x(t)|^2\,dt=\underset{T\to\infty}{\mathrm{Lt}}\int_{-T/2}^{T/2}|e^{-2t}u(t)|^2\,dt$$

$$E=\underset{T\to\infty}{\mathrm{Lt}}\int_{0}^{T/2}\left|e^{-2t}\right|^2\,dt=\underset{T\to\infty}{\mathrm{Lt}}\frac{-1}{4}\left[e^{-4t}\right]_0^{T/2}$$

$$E=\underset{T\to\infty}{\mathrm{Lt}}\frac{-1}{4}\left(e^{-2T}-e^{-0}\right)=\underset{T\to\infty}{\mathrm{Lt}}\frac{-1}{4}\left(e^{-2T}-1\right)$$

$$E_\infty=\frac{-1}{4}(0-1)=\frac{1}{4}<\infty$$

The given signal is an energy signal as energy at ∞ is finite.

$$P=\underset{T\to\infty}{\mathrm{Lt}}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt$$

$$P=\underset{T\to\infty}{\mathrm{Lt}}\frac{1}{T}\int_{-T/2}^{T/2}\left|e^{-2t}u(t)\right|^2\,dt=\underset{T\to\infty}{\mathrm{Lt}}\frac{1}{T}\int_{0}^{T/2}\left|e^{-2t}\right|^2\,dt$$

$$P = \operatorname*{Lt}_{T\to\infty} \frac{1}{T}\int_0^{T/2} \left(e^{-4t}\right) dt = \operatorname*{Lt}_{T\to\infty} \frac{1}{4T}\left(e^{-4t}\right)_0^{T/2}$$

$$P = \operatorname*{Lt}_{T\to\infty} \frac{1}{4T}\left(e^{-2T} - e^{0}\right)$$

$$P_\infty = 0$$

The given signal is not a power signal as power at ∞ is zero.

Problem 2.26 Test whether the given signal is an energy signal or a power signal.

$$x(n) = (-0.5)^n u(n)$$

Solution

$$E_\infty = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} \left|(-0.5)^n u(n)\right|^2$$

$$E_\infty = \sum_{n=0}^{\infty} (0.5)^{2n} = \sum_{n=0}^{\infty} (0.25)^n$$

$$E_\infty = \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

Hint $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

The given signal is an energy signal as E_∞ is finite.

$$P = \operatorname*{Lt}_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} [x(n)]^2$$

$$P = \operatorname*{Lt}_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|(-0.5)^n u(n)\right|^2$$

$$P = \operatorname*{Lt}_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25)^n$$

$$P_\infty = \frac{1}{\infty} = 0$$

The given signal is not a power signal as P_∞ is zero.

Problem 2.27 Test whether the given signal is an energy signal or a power signal.

$$x(n) = u(n)$$

Solution

$$E = \operatorname*{Lt}_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$E = \operatorname*{Lt}_{N\to\infty} \sum_{n=-N}^{N} |u(n)|^2 = \operatorname*{Lt}_{N\to\infty} \sum_{n=0}^{N} 1 = \operatorname*{Lt}_{N\to\infty} (N+1)$$

$$E_\infty = \infty$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |u(n)|^2$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} 1 = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1}(N+1)$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{N\left(1+\dfrac{1}{N}\right)}{N\left(2+\dfrac{1}{N}\right)}$$

$$P_\infty = \frac{1+\dfrac{1}{\infty}}{2+\dfrac{1}{\infty}} = \frac{1+0}{2+0} = \frac{1}{2} < \infty$$

Hence, the signal $x(n) = u(n)$ is a power signal as P_∞ is finite.

Problem 2.28 Test whether the given signal is an energy signal or a power signal.

$$x(t) = tu(t)$$

Solution

$$E = \underset{T\to\infty}{\text{Lt}} \int_{-T/2}^{T/2} |x(t)|^2\, dt = \underset{T\to\infty}{\text{Lt}} \int_{-T/2}^{T/2} |t\,u(t)|^2\, dt$$

$$E = \underset{T\to\infty}{\text{Lt}} \int_{0}^{T/2} t^2 dt = \underset{T\to\infty}{\text{Lt}} \left[\frac{t^3}{3}\right]_0^{T/2}$$

$$E = \frac{1}{3} \underset{T\to\infty}{\text{Lt}} \left[\left(\frac{T}{2}\right)^3 - 0\right] = \frac{1}{3} \underset{T\to\infty}{\text{Lt}} \left(\frac{T}{2}\right)^3$$

$$E_\infty = \infty$$

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} |t\,u(t)|^2\, dt = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{0}^{T/2} t^2 dt$$

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \frac{t^3}{3}\Bigg|_0^{T/2} = \underset{T\to\infty}{\text{Lt}} \frac{1}{T}\left[\frac{(T/2)^3}{3}\right] = \underset{T\to\infty}{\text{Lt}} \frac{T^2}{24}$$

$$P_\infty = \infty$$

Hence, $x(t)$ is neither a power signal nor an energy signal as E_∞ and P_∞ are infinite.

Problem 2.29 Test whether the given signal $x(t) = e^{j(2t+\pi/4)}$ is an energy signal or a power signal.

Solution

$$E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{T/2} \left|e^{j(2t+\pi/4)}\right|^2 dt$$

Hint $\left|e^{j\theta}\right| = 1$

$$E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{T/2} 1\, dt$$

$$E = \underset{T\to\infty}{\mathrm{Lt}} \left[t\right]_{-T/2}^{T/2} = \underset{T\to\infty}{\mathrm{Lt}} \left(\frac{T}{2}+\frac{T}{2}\right) = \underset{T\to\infty}{\mathrm{Lt}} (T)$$

$$E_\infty = \infty$$

$$P = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} |e^{j(2+\pi/4)}|^2\, dt$$

$$P = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} 1\, dt = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\, t\Big|_{-T/2}^{T/2}$$

$$P = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\left(\frac{T}{2}+\frac{T}{2}\right)$$

$$P_\infty = 1 < \infty$$

Hence, it is a power signal as P_∞ is finite.

Problem 2.30 Test whether the given signal is an energy signal or a power signal.

$$x(t) = \cos t$$

Solution

$$E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{T/2} |x(t)|^2 dt = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{T/2} |\cos t|^2 dt$$

Hint $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{T/2} \left(\frac{1+\cos 2t}{2}\right) dt = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{T/2} \frac{1}{2} dt + \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{2} \int_{-T/2}^{T/2} \cos 2t\, dt$$

$$E = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{2}\left(\frac{T}{2}+\frac{T}{2}\right) = \underset{T\to\infty}{\mathrm{Lt}} \frac{T}{2} = \infty$$

$$E_\infty = \infty$$

$$P = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} |\cos(t)|^2\, dt$$

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1+\cos 2t}{2} \right) dt$$

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} \frac{1}{2} dt + \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T/2}^{T/2} \cos 2t \, dt$$

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \, t \Big|_{-T/2}^{T/2} + 0 = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \left(\frac{T}{2} + \frac{T}{2} \right)$$

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}(T)$$

$$P_\infty = \frac{1}{2} < \infty$$

The given signal is a power signal as P_∞ is finite.

Problem 2.31 Test whether the given signal is an energy signal or a power signal.

$$x(n) = u(n) - u(n-5)$$

Solution

$$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{+N} |x(n)|^2$$

$$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} |u(n) - u(n-5)|^2$$

$$E = \sum_{n=0}^{4} 1 = 5 < \infty$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{N+1} \sum_{n=-N}^{+N} |x(n)|^2$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{N+1} \sum_{n=-N}^{N} |u(n) - u(n-5)|^2$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{N+1} \sum_{n=0}^{4} 1$$

$$P_\infty = \frac{1}{\infty} = 0$$

The energy of the signal is finite and the power is zero. Hence, the signal is an energy signal.

Problem 2.32 Test whether the given signal is an energy signal or a power signal.

$$x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$$

Solution

$$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} \right|^2$$

Hint $|e^{j\theta}| = 1$

$$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} 1 = \underset{N\to\infty}{\text{Lt}} (2N+1)$$

$$E_\infty = \infty$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j}\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) \right|^2$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 = \underset{N\to\infty}{\text{Lt}} \frac{1}{(2N+1)}(2N+1)$$

$$E_\infty = \infty$$

The given signal is a power signal as P_∞ is finite.

2.2.5 Deterministic Signal and Random Signal

Deterministic signal A deterministic signal (continuous-time or discrete-time) is a signal about which there is certainty with respect to its values at any time. In a deterministic signal, the future values of the signal are predictable. For example, ECG, sinusoidal signal, square wave, train of pulses, etc (Fig. 2.20).

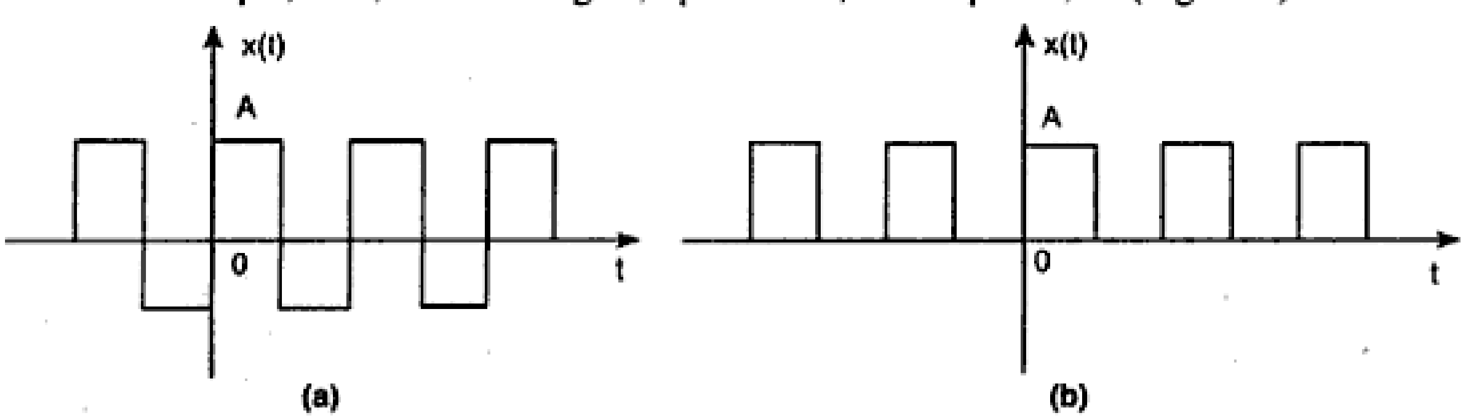


Fig. 2.20 (a) Square Wave (b) Train of Pulses

Random signal A random signal (continuous-time or discrete-time) is a signal about which there is uncertainty with respect to its values at any time. In random signal, the future values of the signal are unpredictable. For example, speech signal, noise, moving object tracking etc (Fig. 2.21).

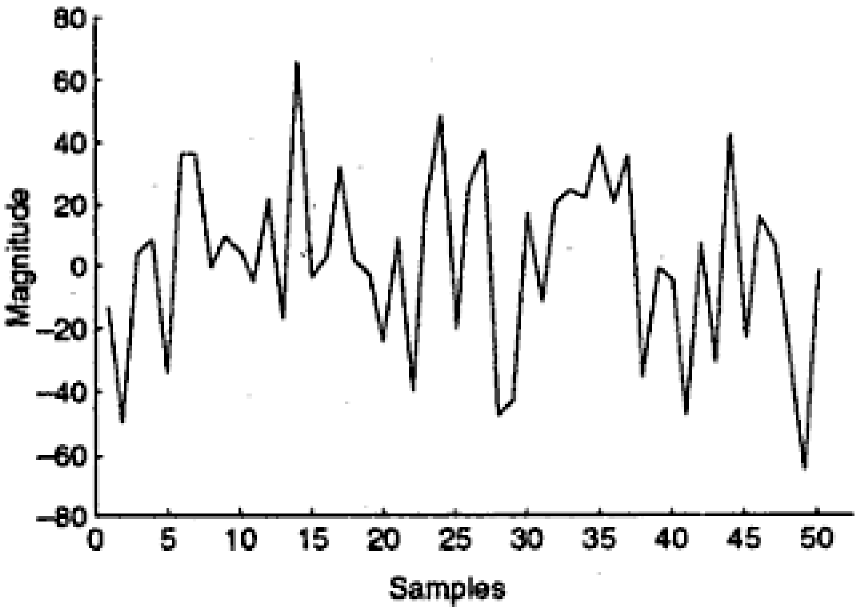


Fig. 2.21 Speech Signal

2.3 BASIC OPERATIONS ON SIGNALS

1. Operations performed on dependent variables
   - Amplitude scaling of signals
   - Addition of signals
   - Multiplication of signals
   - Differentiation on signals
   - Integration on signals

   Any operations listed above, operated on a signal does not affect the period of the signal. It only acts upon the magnitude of the signal.
2. Operations performed on independent variables
   - Time scaling of signals
   - Reflection of signals
   - Time shifting of signals

   Any operations listed above, operated on a signal affects the period of the signal. It does not affect the magnitude of the signal.

2.3.1 Amplitude Scaling of Signals

Consider a continuous-time signal $x(t)$ as shown in Fig. 2.22. Let us introduce an amplitude scaling factor 'α' to the continuous-time signal, i.e.

$$y(t) = \alpha\, x(t) \tag{2.27}$$

where α = scaling factor (if $\alpha < 1$, then the signal attenuates; if $\alpha > 1$, then the signal amplifies).

Let us consider a signal $x(t) = \alpha \sin \omega t$. When $\alpha = 0.5$, the signal reduces to $y_1(t) = 0.5 \sin \omega t$. Only magnitude of the signal reduced, whereas the period of the signal unchanged as shown in Fig. 2.22(a).

When $\alpha = 1.5$, the signal $x(t)$ become $y_2(t) = 1.5 \sin \omega t$. Only magnitude of the signal increased, whereas the period of the signal remain unchanged as shown in Fig. 2.22(b).

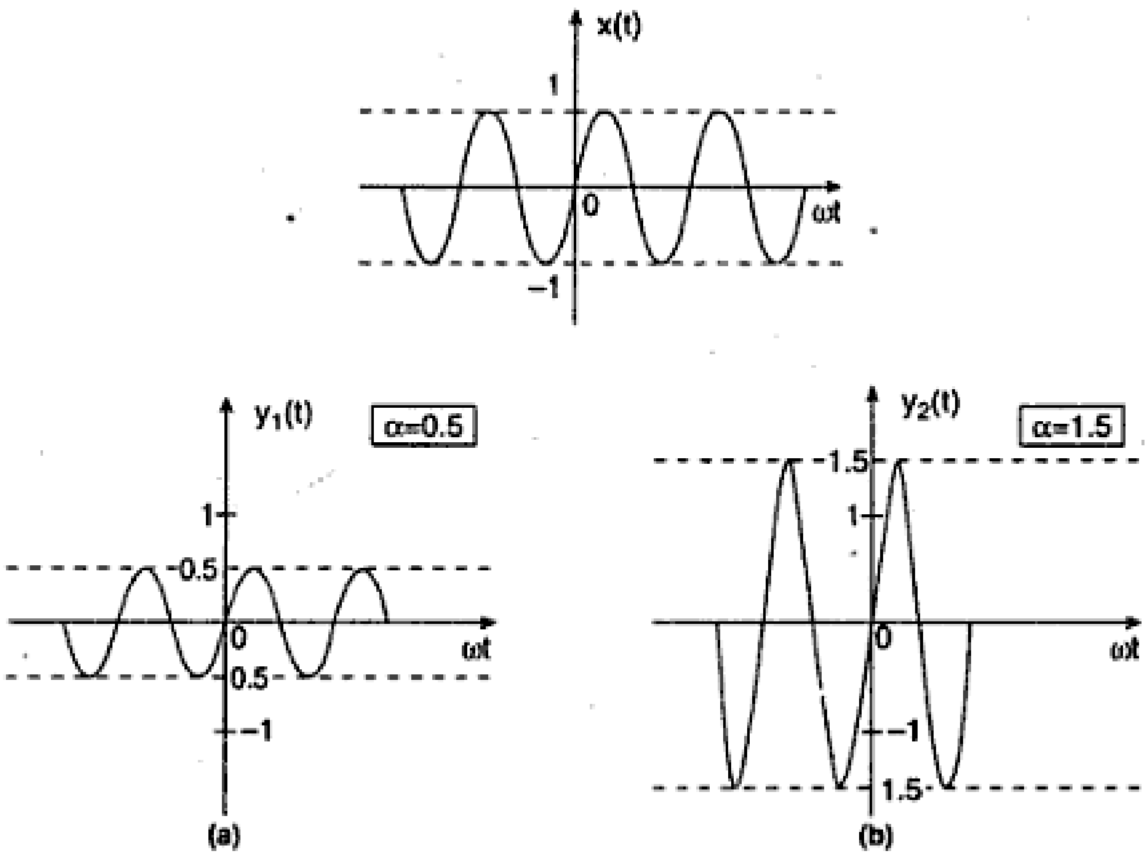


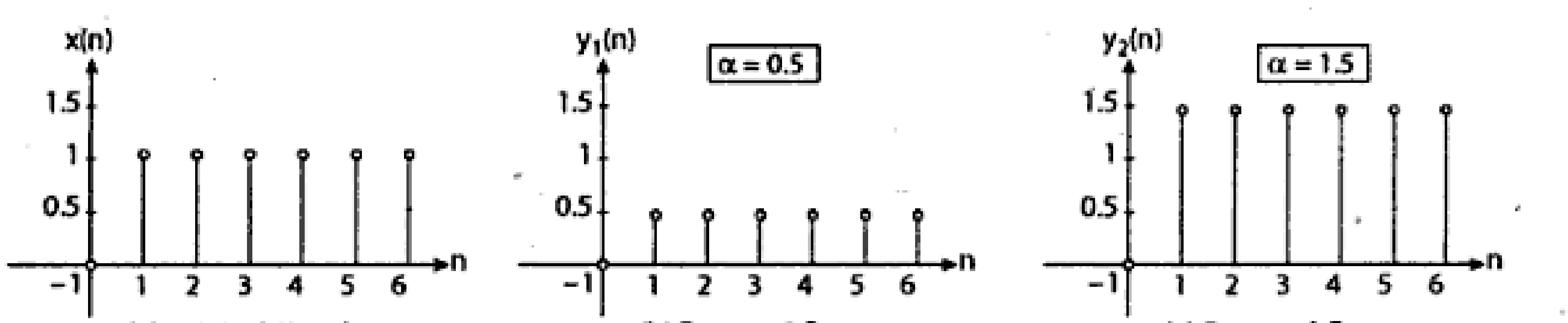
Fig. 2.22 Amplitude Scaling (a) **α<1** ***(b)*** **α>1**

Consider a discrete-time signal $x(n)$as shown in Fig. 2.23. Let us introduce an amplitude scaling factor α to the discrete-time signal, i.e.

$$y(n) = \alpha x(n) \tag{2.28}$$

where, α = scaling factor (if $\alpha < 1$, then the signal attenuates; if $\alpha > 1$, then the signal amplifies).

Let us consider $x(n) = u(n),\ 5 \geq n \geq 0$. When $\alpha = 0.5$, $y_1 = 0.5u(n)$. Only magnitude of the signal $y_1(t)$ reduced, whereas the number of samples remain same as shown in Fig. 2.23(b). Similarly, when $\alpha = 1.5$, $y_2(n) = 1.5u(n)$. Only magnitude of the signal $y_2(t)$ increased, whereas the number of samples remain same as shown in Fig. 2.23(c).



(a) Original Signal (b) For α = 0.5 (c) For α = 1.5

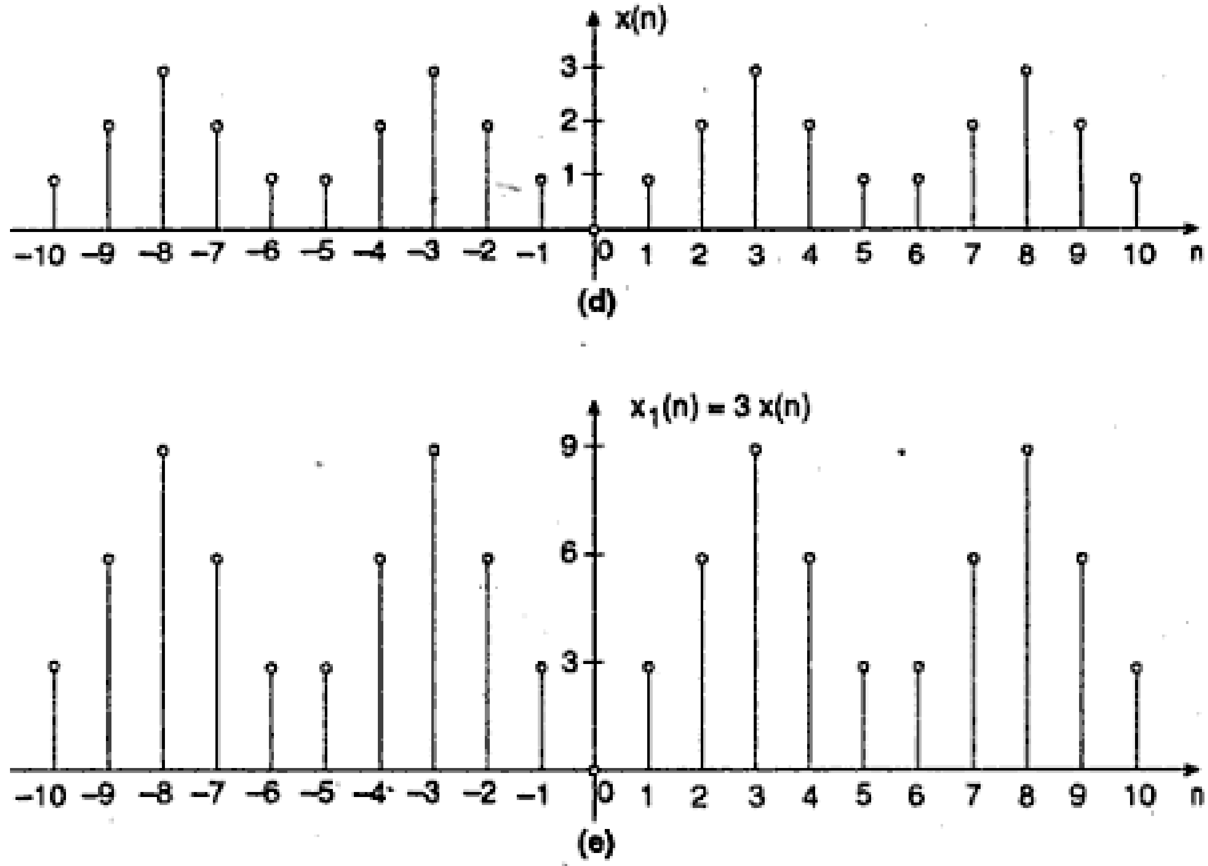
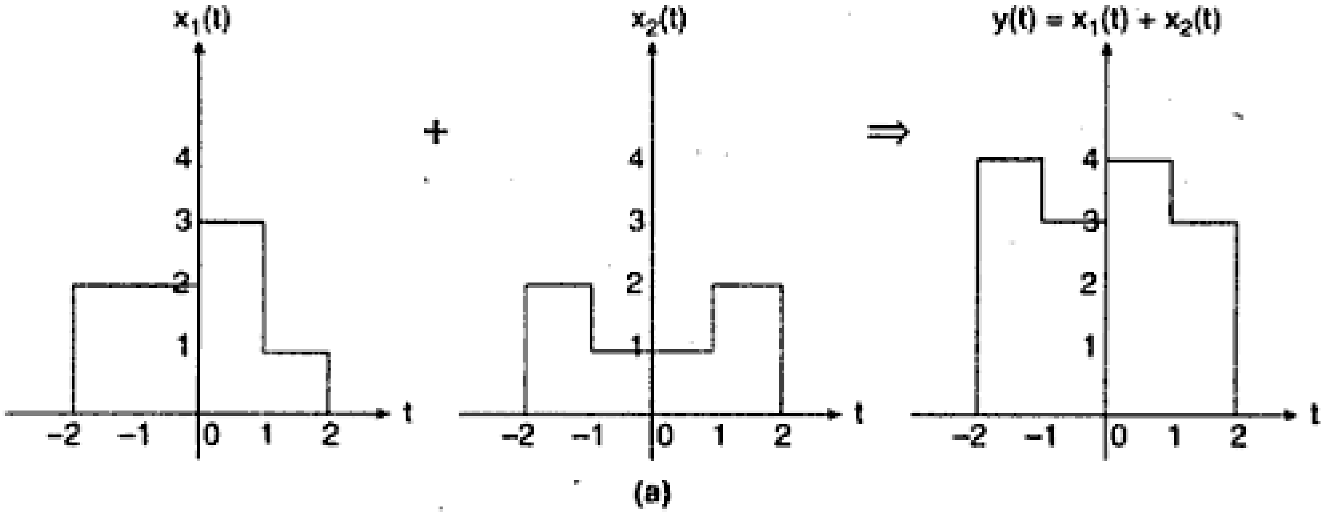


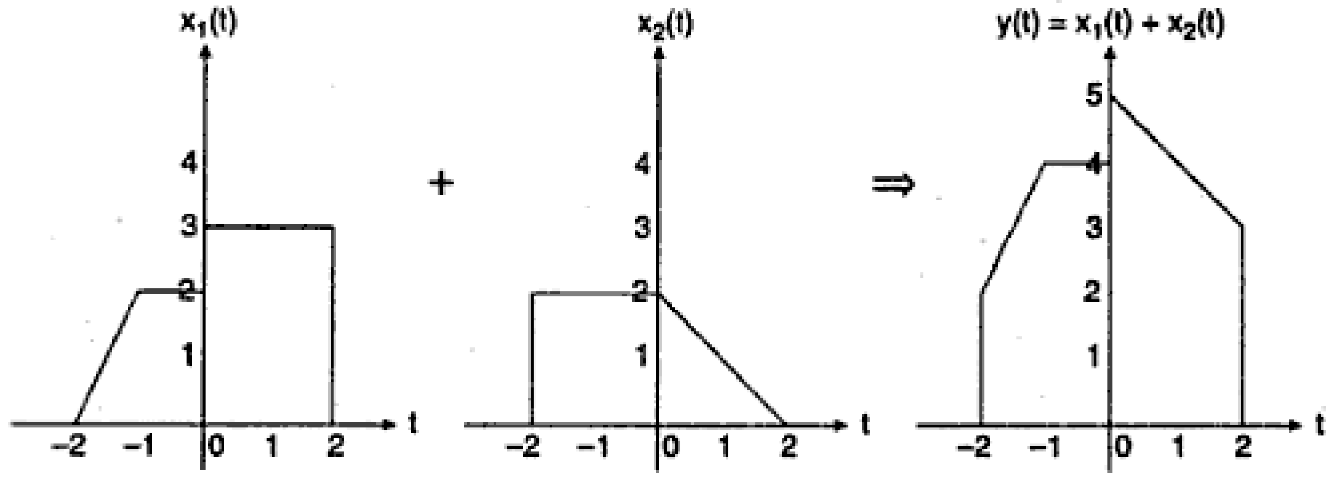
Fig. 2.23 Amplitude Scaling (d) Original Signal (e) Scaled Signal (α=3)

2.3.2 Addition of Signals

Consider a pair of continuous-time signals $x_1(t)$ and $x_2(t)$ as shown in Fig. 2.24. Adding these two signals, $x_1(t)$ and $x_2(t)$, results in a signal $y(t)$. It is important to note that the period of the output signal is unaltered.

$$y(t) = x_1(t) + x_2(t) \tag{2.29}$$



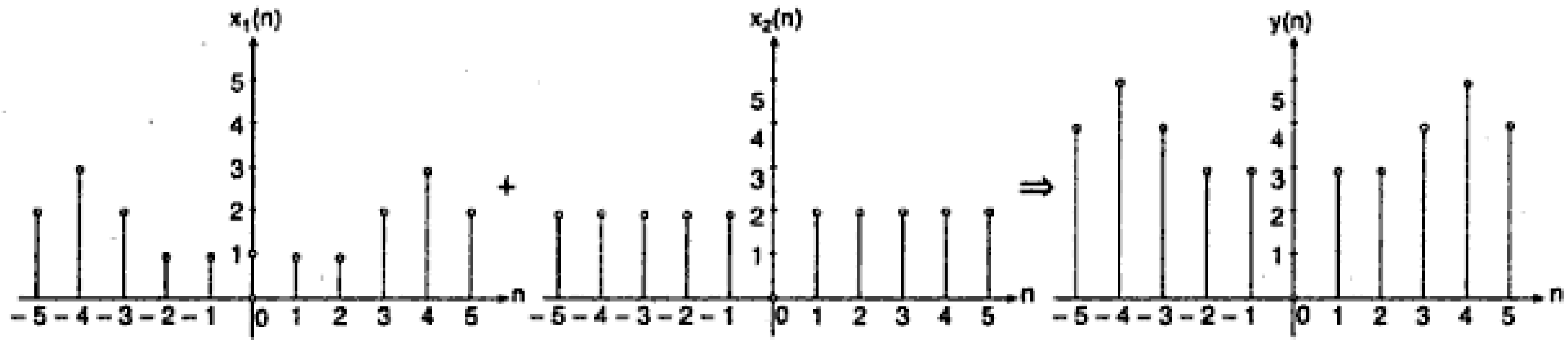


(b)

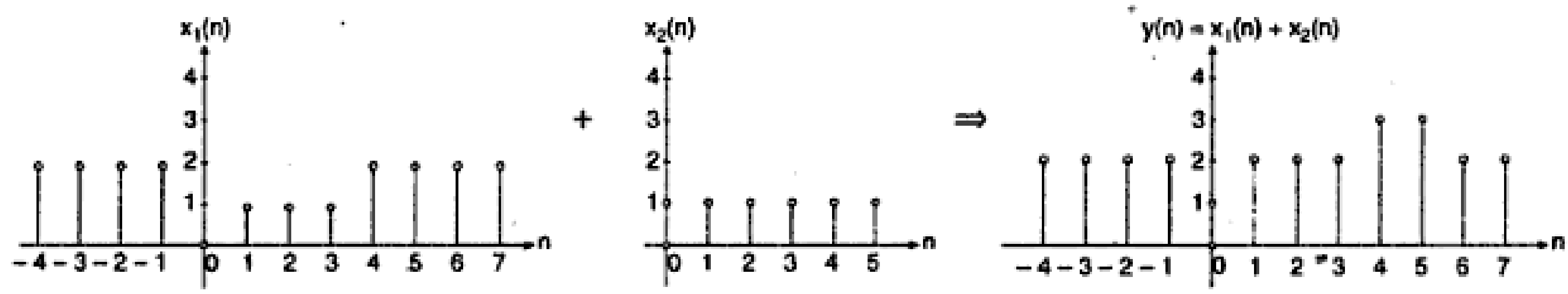
Fig. 2.24 Addition of Signals

Consider a pair of discrete-time signals $x_1(n)$ and $x_2(n)$ as shown in Fig. 2.25. Adding these two signals, $x_1(n)$ and $x_2(n)$, results in a signal $y(n)$. The period of $y(n)$ is unchanged.

$$y(n) = x_1(n) + x_2(n) \tag{2.30}$$



(a)



(b)

Fig. 2.25 Addition of Signals

2.3.3 Multiplication of Signals

Consider a pair of continuous-time signals, $x_1(t)$ and $x_2(t)$as shown in Fig. 2.26. Multiplication of these two signals, $x_1(t)$ and $x_2(t)$, results in a signal $y(t)$, i.e.

$$y(t) = x_1(t) \times x_2(t) \tag{2.31}$$

Multiplication of message signal $x_1(t)$ over the carrier signal $x_2(t)$ results in a modulated signal, which is used to transmit over communication media.

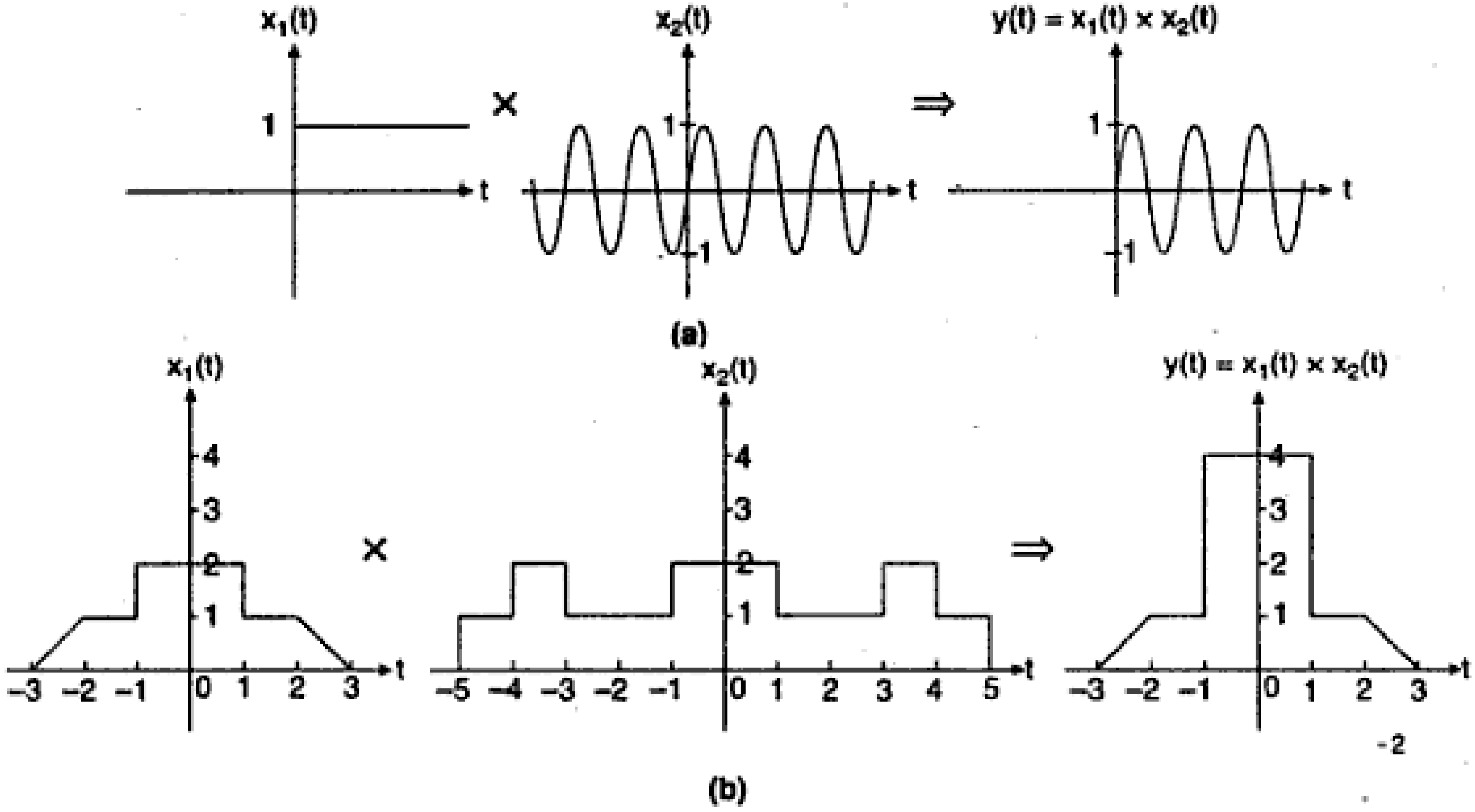


Fig. 2.26 Multiplication of Signals

Consider a pair of discrete-time signal $x_1(n)$ and $x_2(n)$ as shown in Fig. 2.27. Multiplication of these two discrete-time signals, $x_1(n)$ and $x_2(n)$, results in a output signal $y(n)$, i.e.

$$y(n) = x_1(n) \times x_2(n) \tag{2.32}$$

2.3.4 Differentiation on Signals

The derivative of an input signal $x(t)$ with respect to time is defined by

$$y(t) = \frac{d\,x(t)}{dt} \tag{2.33}$$

Example The voltage $v(t)$ developed across the inductor is a derivative of the current $i(t)$ flowing through the inductor L, i.e.

$$v(t) = L\frac{d\,i(t)}{dt} \tag{2.34}$$

Differentiation of continuous time signal is equivalent to the difference of the discrete-time signal $x(n)$.

$$x(n) - x(n-1) \tag{2.35}$$

The differentiation of square wave and sinusoidal wave are illustrated in Fig. 2.28 (a) and Fig. 2.28 (b) respectively.

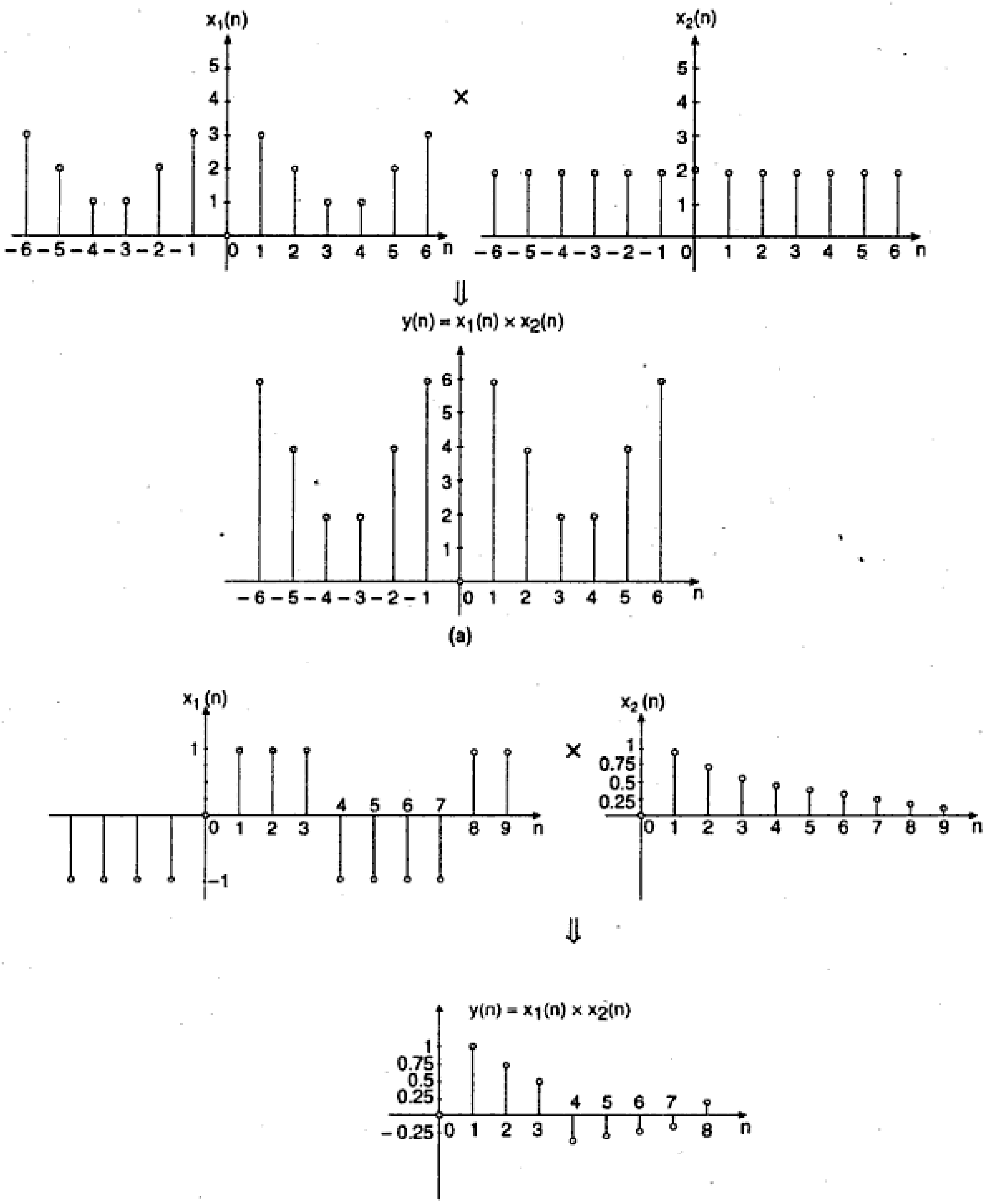


Fig. 2.27 Multiplication of Signals

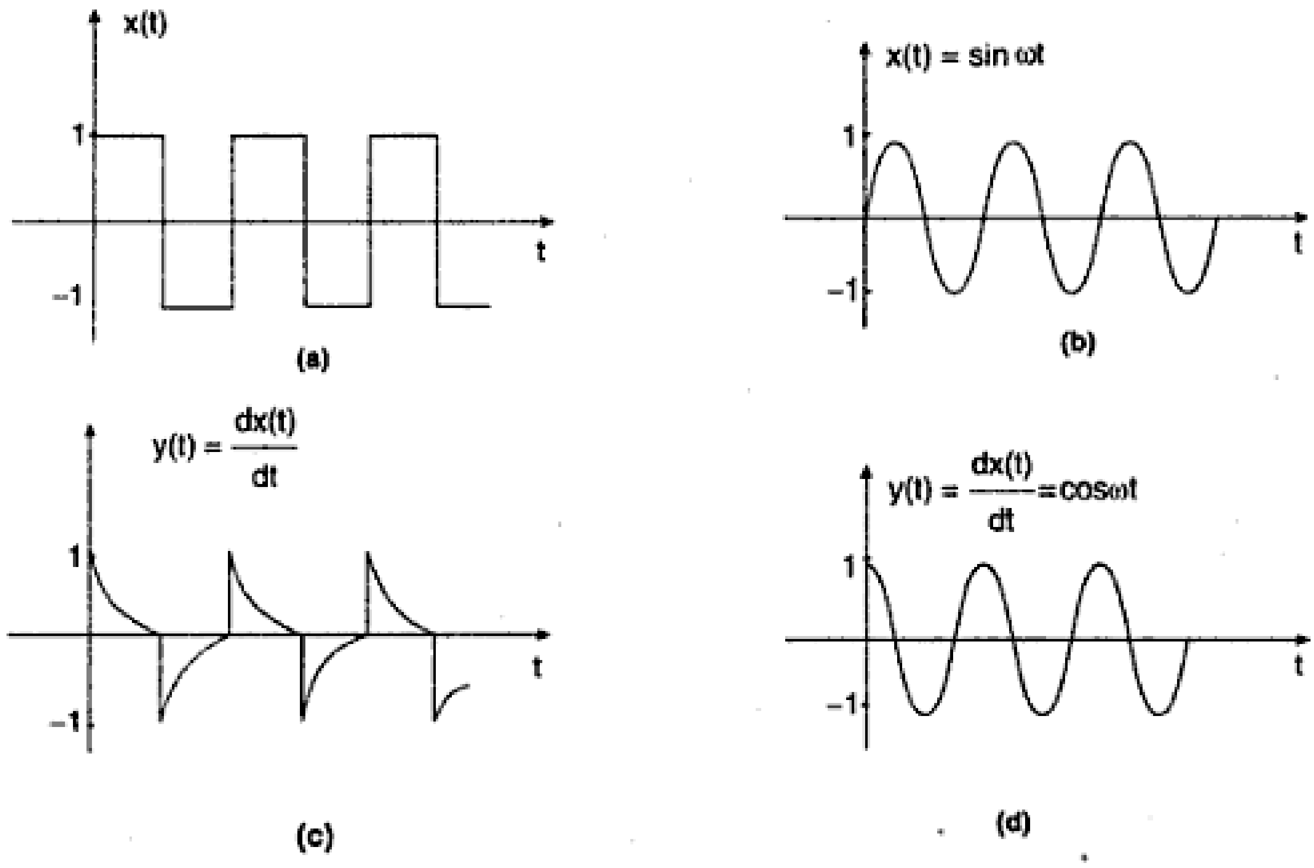


Fig. 2.28 Differentiation (a) Square Wave (b) Sinusoidal Wave (c) Differentiated Square Wave (d) Differentiated Sinusoidal Wave

2.3.5 Integration on Signals

The integration of an input signal $x(t)$ with respect to time $y(t)$ is defined by

$$y(t) = \int_{-\infty}^{t} x(t)\, dt$$

Example The voltage $v(t)$ developed across the capacitor is a derivative of the current $i(t)$ flowing through the capacitor C. i.e.

$$y(t) = \frac{1}{C} \int_{-\infty}^{t} i(t)\, dt \tag{2.36}$$

The integration of a continuous-time signal $x(t)$ is equivalent to summation of discrete-time signal $x(n)$.

$$y(n) = \sum_{n} x(n) \tag{2.37}$$

The integration of square wave and cosinusoidal wave are illustrated in Fig. 2.29 (a) and Fig. 2.29 (b) respectively.

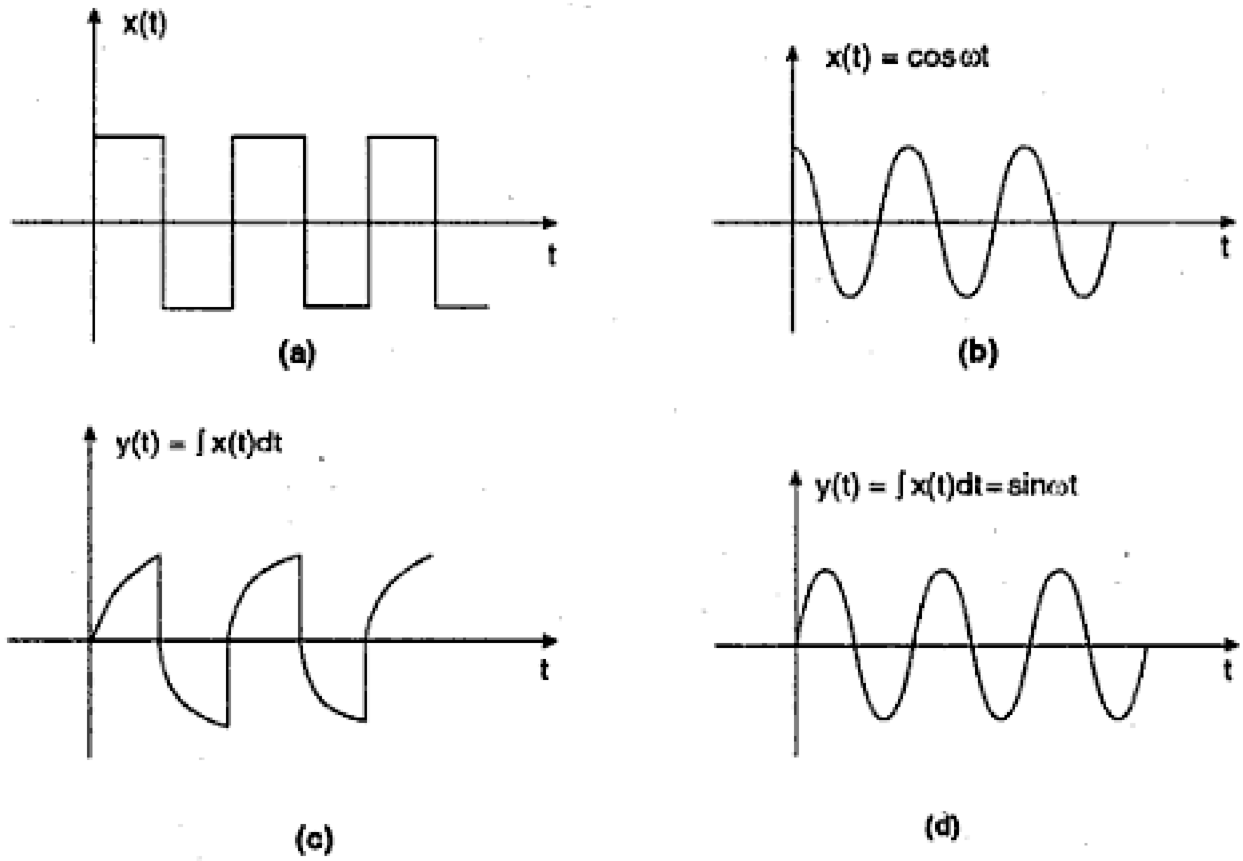


Fig. 2.29 Integration (a) Square Wave (b) Cosinusoidal Wave (c) Integrated Square Wave (d) Integrated Cosinusoidal Wave

2.3.6 Time Scaling of Signals

Consider a continuous-time signal $x(t)$ as shown in Fig. 2.30. Let us introduce a time scaling factor β to the continuous-time signal, i.e.

$$y(t) = x(\beta t) \tag{2.38}$$

where β = scaling factor (if $\beta < 1$, then the signal expands; if $\beta > 1$, then the signal compresses).

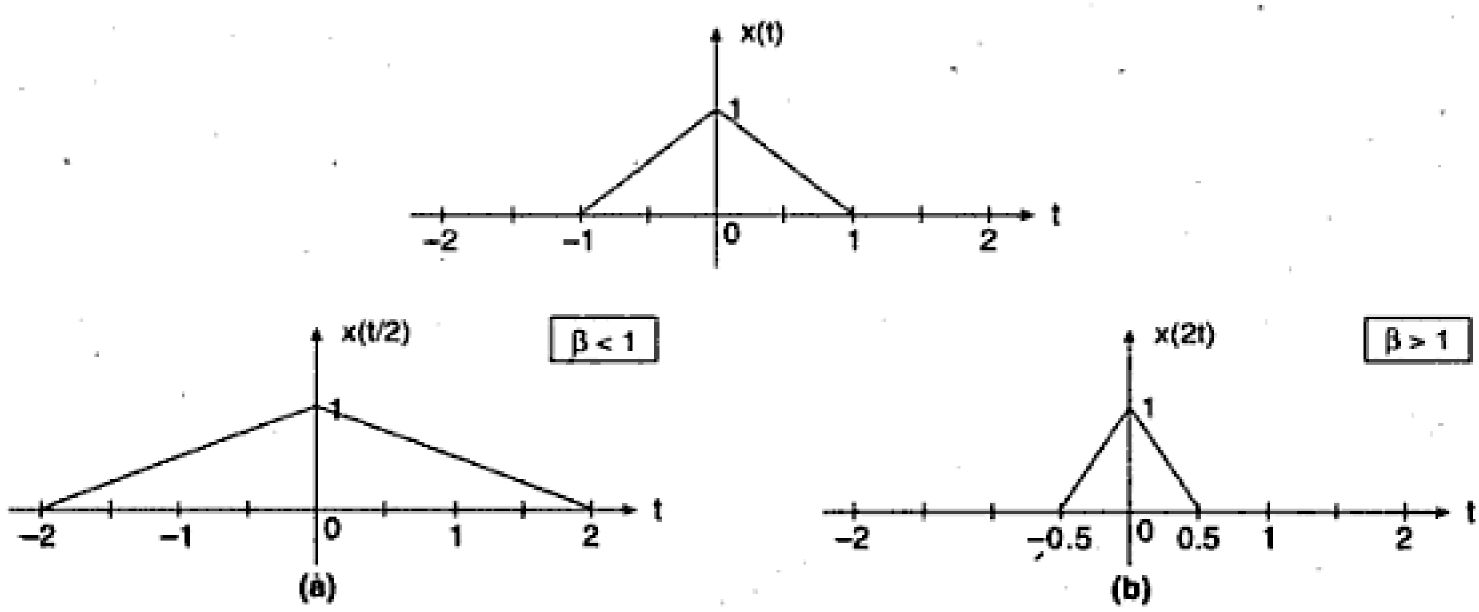


Fig. 2.30 Time Scaling (a) $\beta < 1$ (b) $\beta > 1$

Consider a discrete-time signal $x(n)$ as shown in Fig. 2.30. Let us introduce a time scaling factor β to the discrete-time signal, i.e.

$$y(t) = x(\beta n) \tag{2.39}$$

where β = scaling factor (if $\beta < 1$, then the signal expands; if $\beta > 1$, then the signal compresses.)

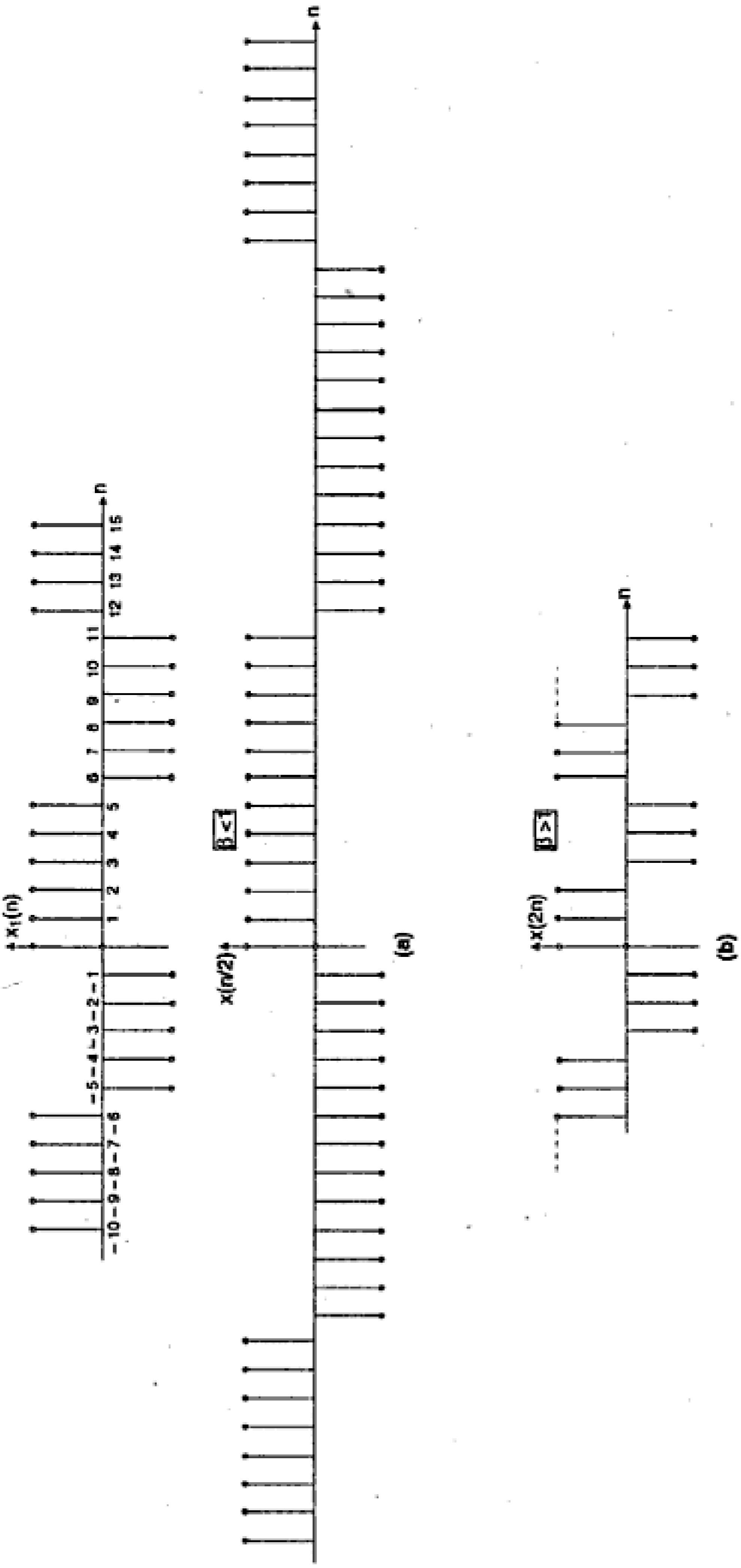


Fig. 2.31 Time Scaling (a) $\beta < 1$ *(b)* $\beta > 1$

2.3.7 Reflection of Signals

Consider a continuous-time signal $x(t)$ as shown in Fig. 2.32(a). Let $y(t)$ denote a signal obtained by replacing t by $-t$ (time-reflection) to the continuous-time signal, i.e.

$$y(t) = x(-t) \tag{2.40}$$

If $x(-t) = x(t)$ for all values of t, then the reflected signal is an even signal, or if $x(-t) = -x(t)$ for all values of t, then the reflected signal is an odd signal.

Consider a discrete-time signal $x(n)$ as shown in Fig. 2.32(b). Let $y(n)$ denote a signal obtained by replacing n by $-n$ (time-reflection) to the discrete-time signal, i.e.

$$y(n) = x(-n) \tag{2.41}$$

If $x(-n) = x(n)$ for all values of n, then the reflected signal is an even signal, or if $x(-n) = -x(n)$ for all values of n, then the reflected signal is an odd signal.

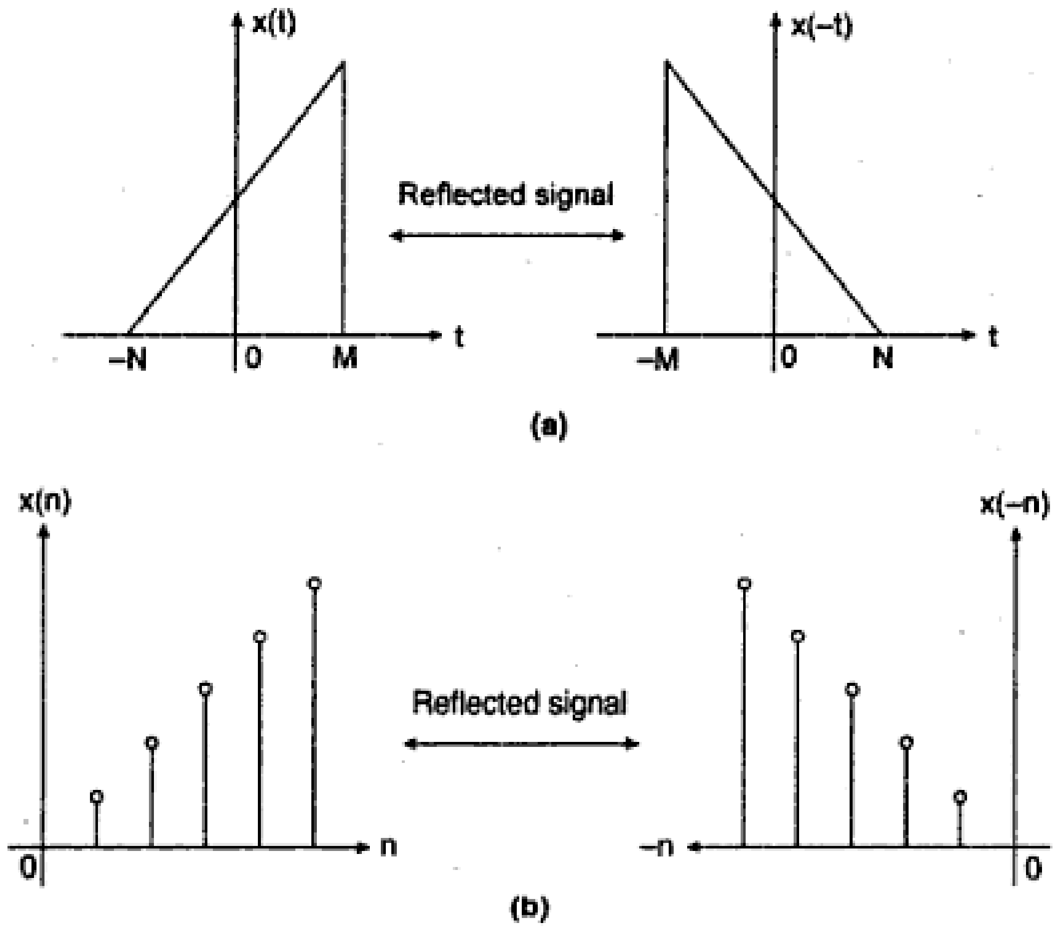


Fig. 2.32 Reflection of Signals (a) Continuous-time Signal (b) Discrete-time Signal

2.3.8 Time Shifting of Signals

Consider a continuous-time signal $x(t)$. Let $y(t)$ denote a signal obtained by shifting the signal $x(t)$ by $(t - t_0)$, that is,

$$y(t) = x(t - t_0) \tag{2.42}$$

If the signal $x(t)$ is positive, and $t_0 > 0$ for all values of t_0, then the signal is said to be right-shifted signal. In the example shown in Fig. 2.33(b), the signal is shifted to right side by 3 units.

If the signal $x(t)$, is positive, and $t_0 < 0$ for all values of t_0 then the signal is said to be left-shifted signal. In the example shown in Fig. 2.33(c), the signal is shifted to left side by 4 units.

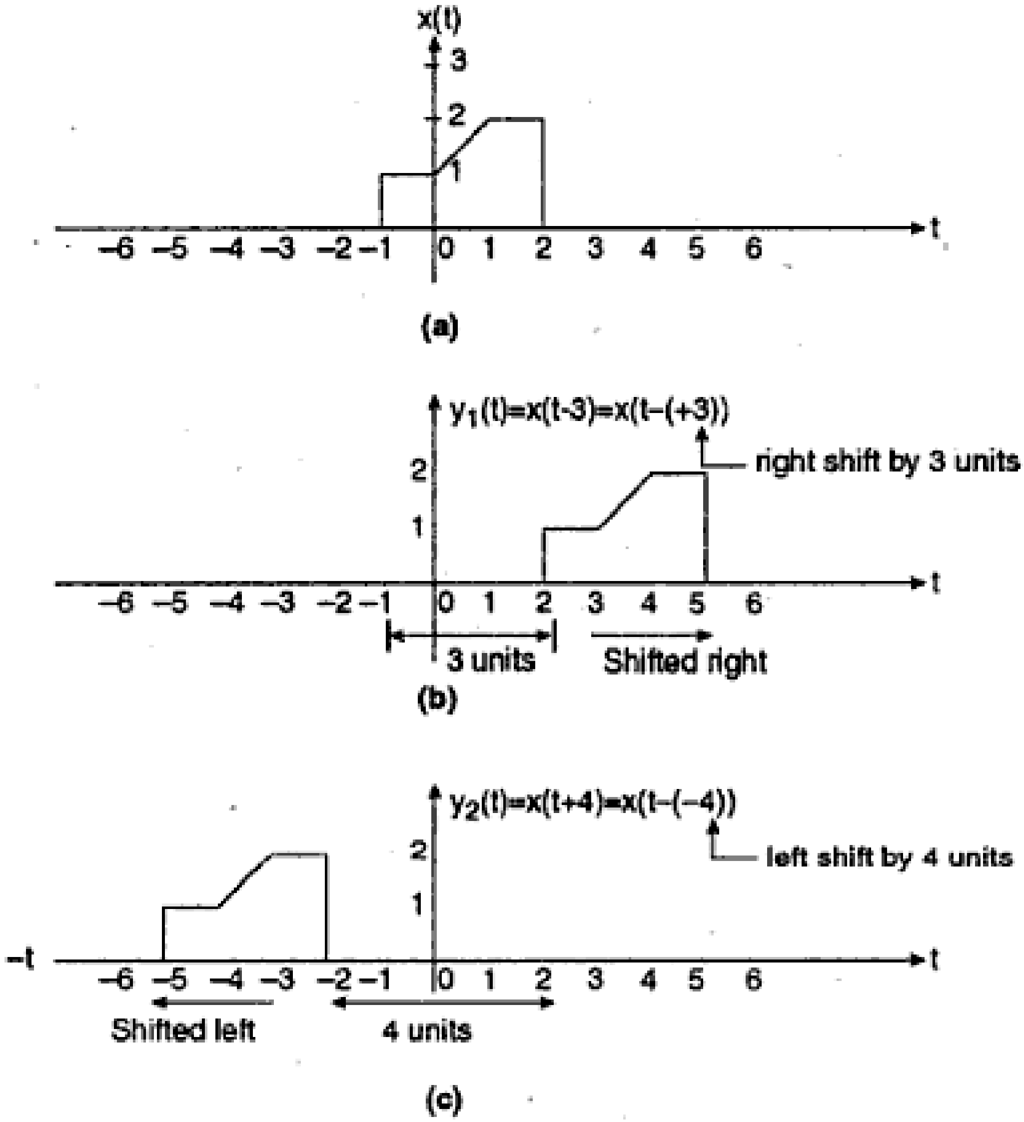
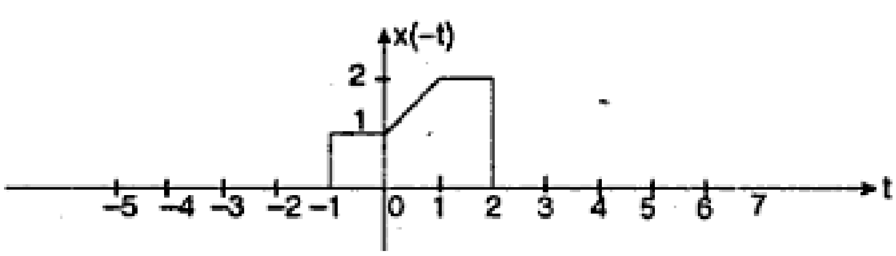


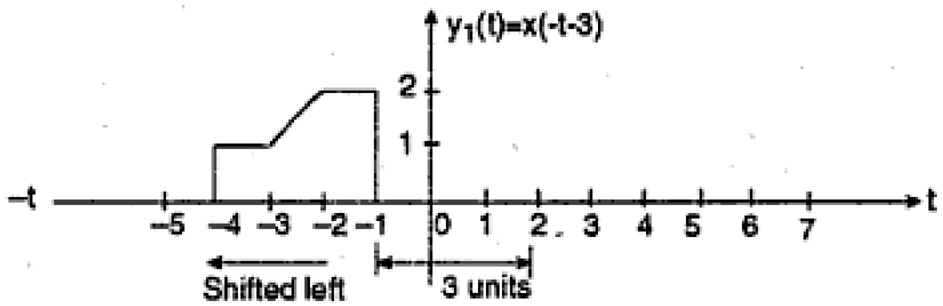
***Fig. 2.33** (a) Original Positive Signal x(t) (b) Right-shifted Signal x(t-3) (c) Left-shifted Signal x(t+4)*

If the signal $x(-t)$ is negative, and $t_0 > 0$ for all values of t_0, then the signal is said to be left-shifted signal. In the example shown in Fig. 2.34(b), the signal is shifted to left-side by 3 units.

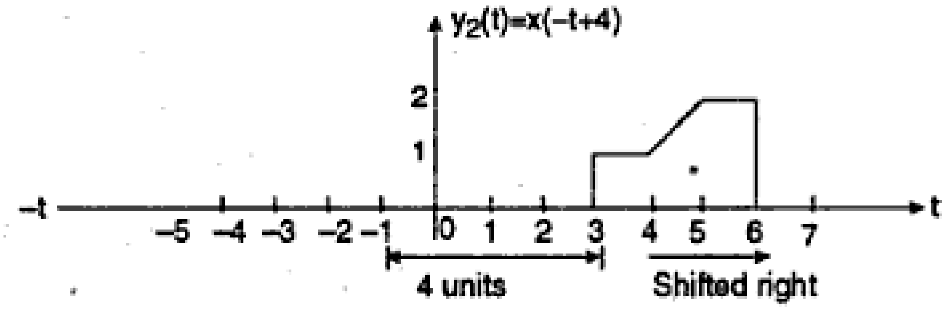
If the signal $x(-t)$ is negative, and $t_0 < 0$ for all values of t_0, then the signal is said to be right-shifted signal. In the example shown in Fig. 2.34(c), the signal is shifted to right-side by 4 units.



(a)



(b)



(c)

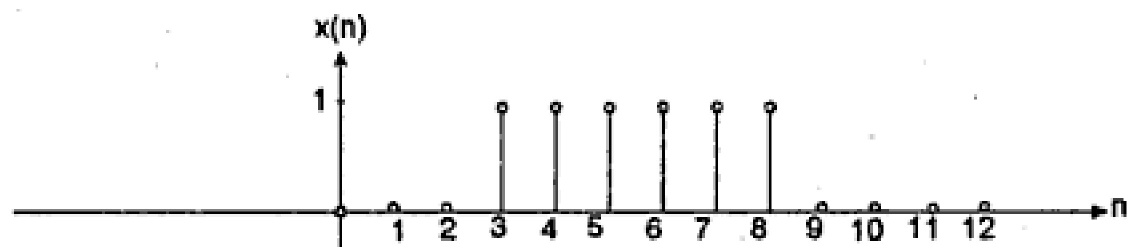
***Fig. 2.34** (a) Original Negative Signal x(-t) (b) Left-shifted Signal x(-t-3) (c) Right-shifted Signal x(-t+4)*

The above analysis also holds good for discrete-time signals. Let us consider a discrete-time signal $x(n)$. Let $y(n)$ denote a signal obtained by shifting the signal $x(n)$ by $(n-n_0)$, that is

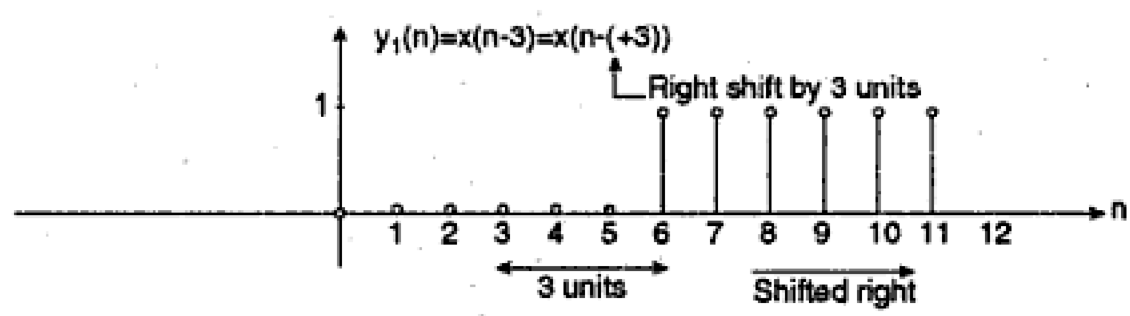
$$y(n) = x(n-n_0) \tag{2.44}$$

If the signal $x(n)$ is positive, and $n_0 > 0$ for all values of n_0, then the signal is said to be right-shifted signal. In the example, shown in Fig. 2.35(b), the signal is shifted to right side by 3 units.

If the signal $x(n)$ is positive, and $n_0 < 0$ for all values of n_0, then the signal is said to be left-shifted signal. In the example, shown in Fig. 2.35(c), the signal is shifted to left side by 4 units.



(a)



(b)

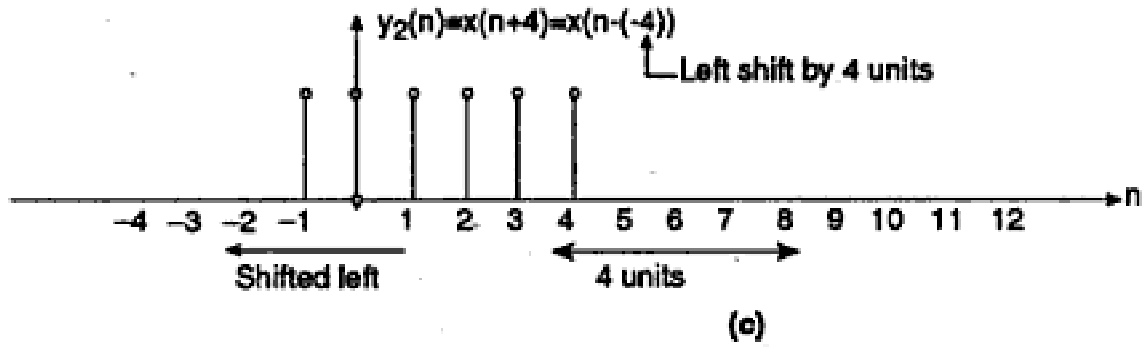


Fig. 2.35 (a) Original Positive Signal x(n) (b) Right-shifted Signal x(n-3) (c) Left-shifted Signal x(n+4)

If the signal $x(-n)$ is negative, and $n_0 > 0$ for all values of n_0 then the signal is said to be right-shifted signal. In the example shown in Fig. 2.36(b), the signal is shifted to left side by 3 units.

If the signal $x(-n)$ is negative, and $n_0 < 0$ for all values of n_0 then the signal is said to be left-shifted signal. In the example shown in Fig. 2.36(c), the signal is shifted to right side by 4 units.

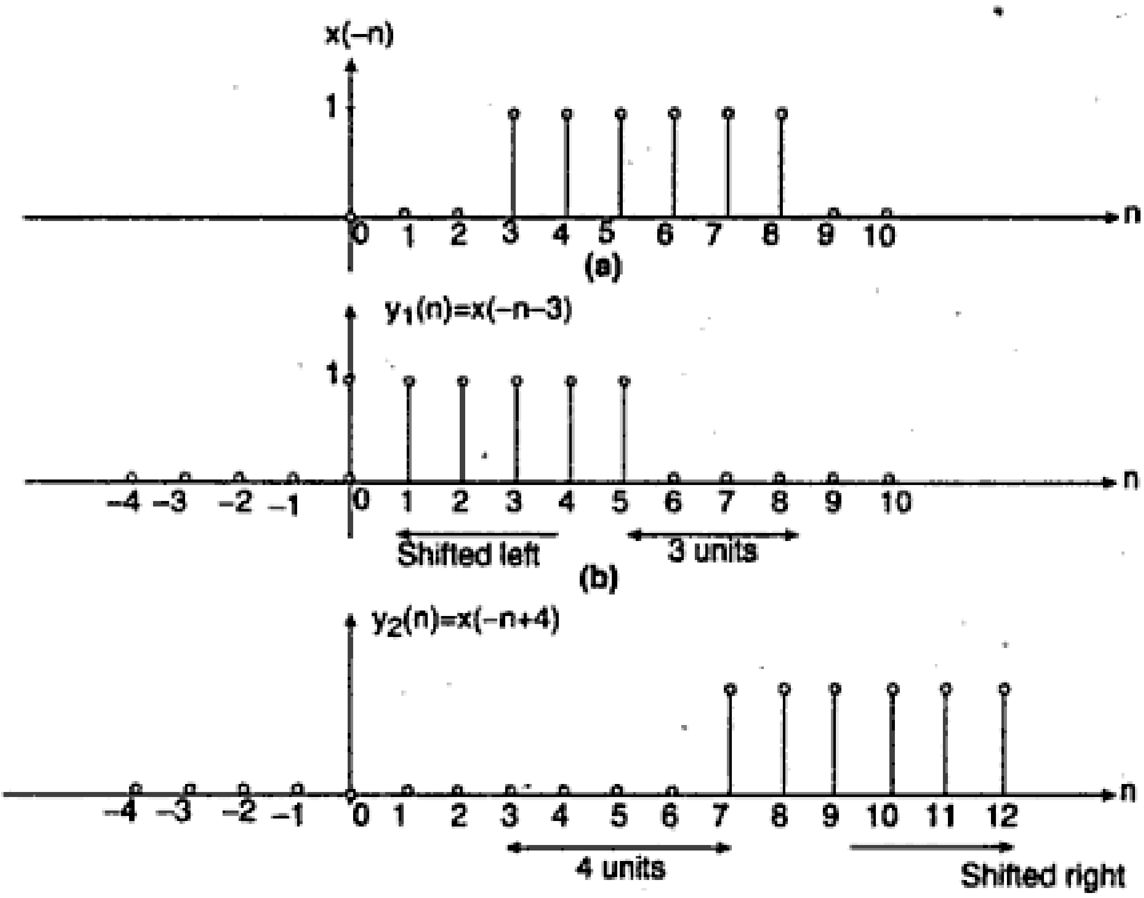


Fig. 2.36 (a) Original Negative Signal x(−n) (b) Left-shifted Signal x(−n−3) (c) Right-shifted Signal x(−n+4)

2.3.9 Time Shifting and Time Scaling

The time shifting operation is performed first on $x(t)$ resulting in an intermediate signal $v(t)$.

$$v(t) = x(t - b) \tag{2.46}$$

Next, the time scaling operation is performed on $v(t)$. This replaces t by at resulting in the desired output.

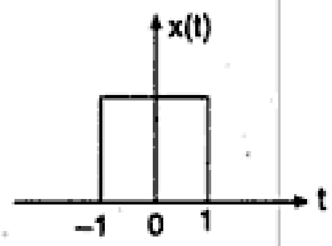
$$y(t) = v(at) = x(at - b) \tag{2.47}$$

SOLVED PROBLEMS

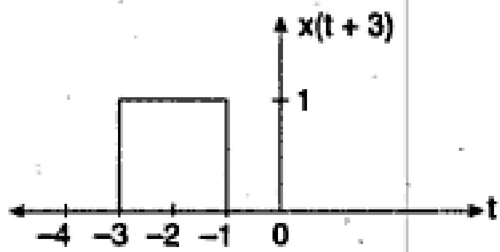
Problem 2.33 (i) Find $x(2t+3)$ for a given signal $x(t)$.

Solution

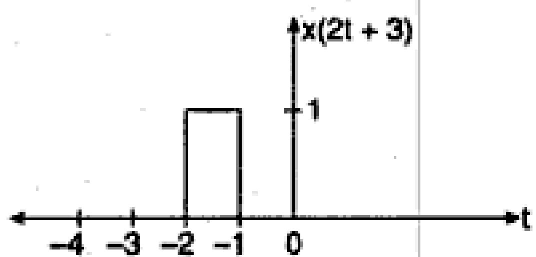
Step (i) $x(t+3) = x[t-(-3)]$



Step (ii) Time shift $x(t+3)$



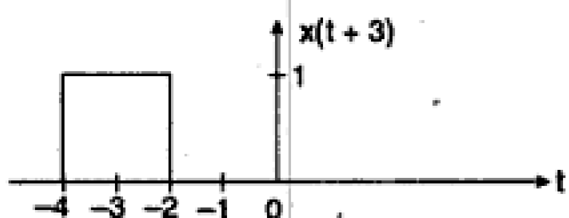
Step (iii) Time scale $x(2t+3)$



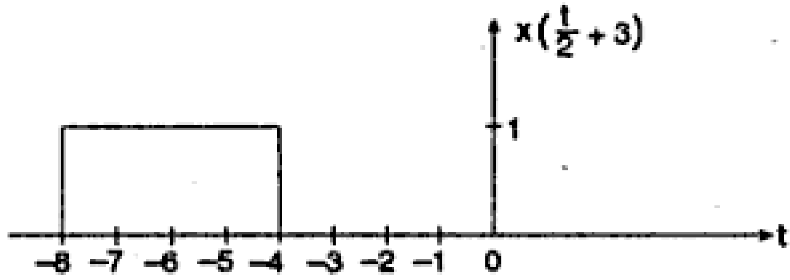
Lower Bound	Upper Bound
$2t+3=-1$	$2t+3=\ \ 1$
$2t=-4$	$2t=-2$
$t=-2$	$t=-1$

(ii) Find $y(t) = u\left(\dfrac{t}{2}+3\right)$ for the signal given in Problem 2.32 (a).

Step (i) $x(t+3)$



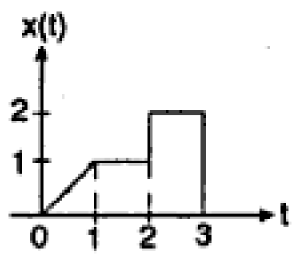
Step (ii) $u\left(\frac{t}{2}+3\right)$



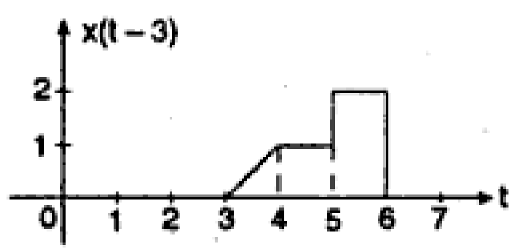
Lower Bound	Upper Bound
$t/2 + 3 = -1$	$t/2 + 3 = 1$
$t/2 = -4$	$t/2 = -2$
$t = -8$	$t = -4$

Problem 2.34 Find $x(2t-3)$ for the given signal $x(t)$.

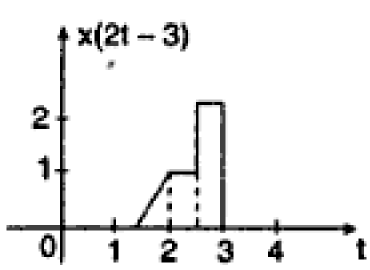
Solution
Step (i)



Step (ii)



Step (iii)

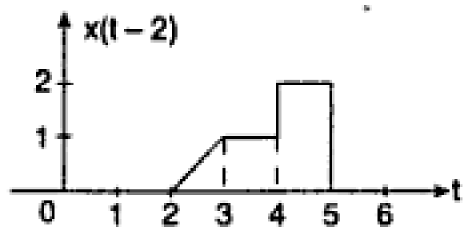


Bound I		Bound II		Bound III	
Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
$2t-3=0$ $2t=3$ $t=1.5$	$2t-3=1$ $2t=4$ $t=2$	$2t-3=1$ $2t=4$ $t=2$	$2t-3=2$ $2t=5$ $t=2.5$	$2t-3=2$ $2t=5$ $t=2.5$	$2t-3=3$ $2t=6$ $t=3$

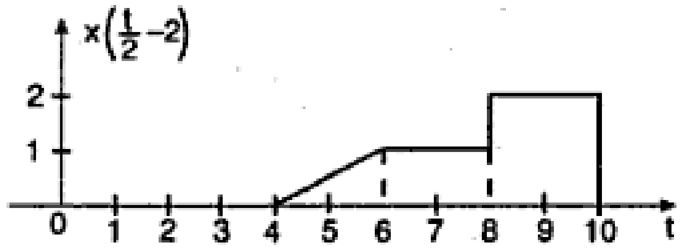
Problem 2.35 Find $x\left(\frac{t-4}{2}\right)$ for the signal given in Problem 2.33.

Solution

Step (i)



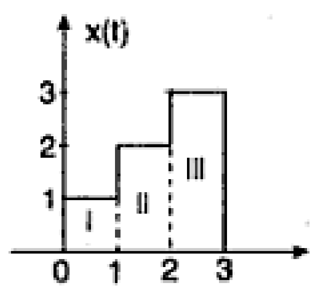
Step (ii)



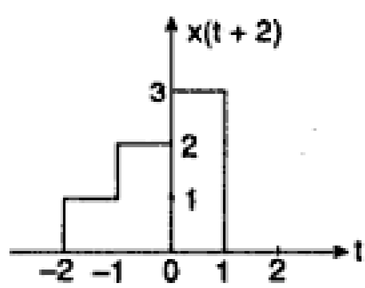
Bound I		Bound II		Bound III	
Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
$t/2-2=0$	$t/2-2=1$	$t/2-2=1$	$t/2-2=2$	$t/2-2=2$	$t/2-2=3$
$t/2=2$	$t/2=3$	$t/2=3$	$t/2=4$	$t/2=4$	$t/2=5$
$t=4$	$t=6$	$t=6$	$t=8$	$t=8$	$t=10$

Problem 2.36 Find $x(3t+2)$ and $x\left(\frac{t}{3}+2\right)$ for the given signal $x(t)$.

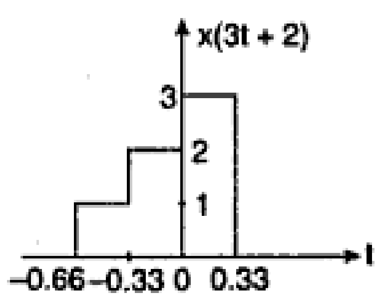
Step (i)



Step (ii)

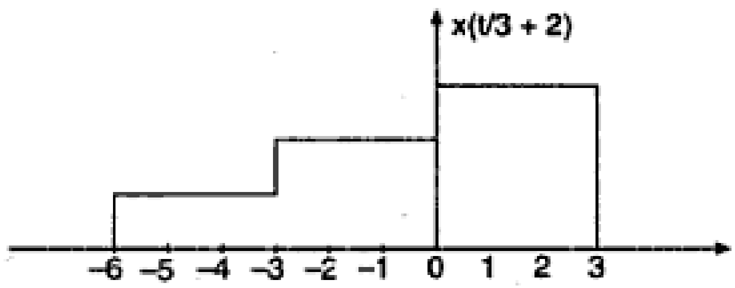


Step (iii)



Bound I		Bound II		Bound III	
Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
$3t+2=0$ $3t=-2$ $t=-2/3$	$3t+2=1$ $3t=-1$ $t=-1/3$	$3t+2=1$ $3t=-1$ $t=-1/3$	$3t+2=2$ $3t=0$ $t=0$	$3t+2=2$ $3t=0$ $t=0$	$3t+2=3$ $3t=1$ $t=1/3$

Step (iv)



Bound I		Bound II		Bound III	
Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
$t/3+2=0$ $t/3=-2$ $t=-6$	$t/3+2=1$ $t/3=-1$ $t=-3$	$t/3+2=1$ $t/3=-1$ $t=-3$	$t/3+2=2$ $t/3=0$ $t=0$	$t/3+2=2$ $t/3=0$ $t=0$	$t/3+2=3$ $t/3=1$ $t=3$

■ 2.4 TYPES OF SIGNALS

In this section, we shall study the various basic signals necessary to construct different signals, which are necessary for the system analysis. The basic signals are as follows:

1. Exponential signal
   (i) Real exponential signal
   (ii) Complex exponential signal
2. Sinusoidal signal
3. Step signal
4. Impulse signal
5. Ramp signal

2.4.1 Exponential Signal

Real exponential signal (continuous-time signal) A real exponential signal $x(t)$ in its most general form is represented by

$$x(t) = B\,e^{at} \tag{2.48}$$

where B is scaling factor, (real parameter) and α is real parameter. Depending on the value of α, a real exponential signal can be further divided into two more signals.

For $\alpha < 0$, the magnitude of a real exponential signal decays (decreases) exponentially and is illustrated in Fig. 2.37 (a).

For $\alpha > 0$, the magnitude of a real exponential signal rises (increases) exponentially and is illustrated in Fig. 2.37 (b).

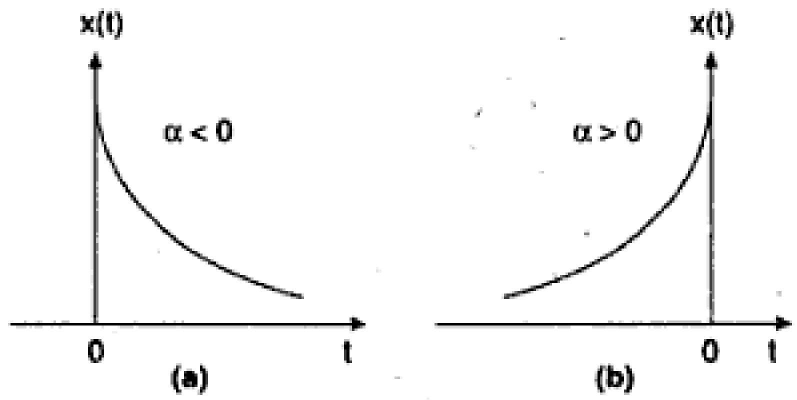


Fig. 2.37 Continuous-time Exponential Signal (a) $\alpha < 0$, (b) $\alpha > 0$

Example Charging of a capacitor is an example of a growing exponential signal, and discharging of a capacitor is an example of a decaying exponential signal.

Real exponential signal (discrete-time signal) A real exponential signal $x(n)$ in its most general form is represented by

$$x(n) = B\alpha^n \tag{2.49}$$

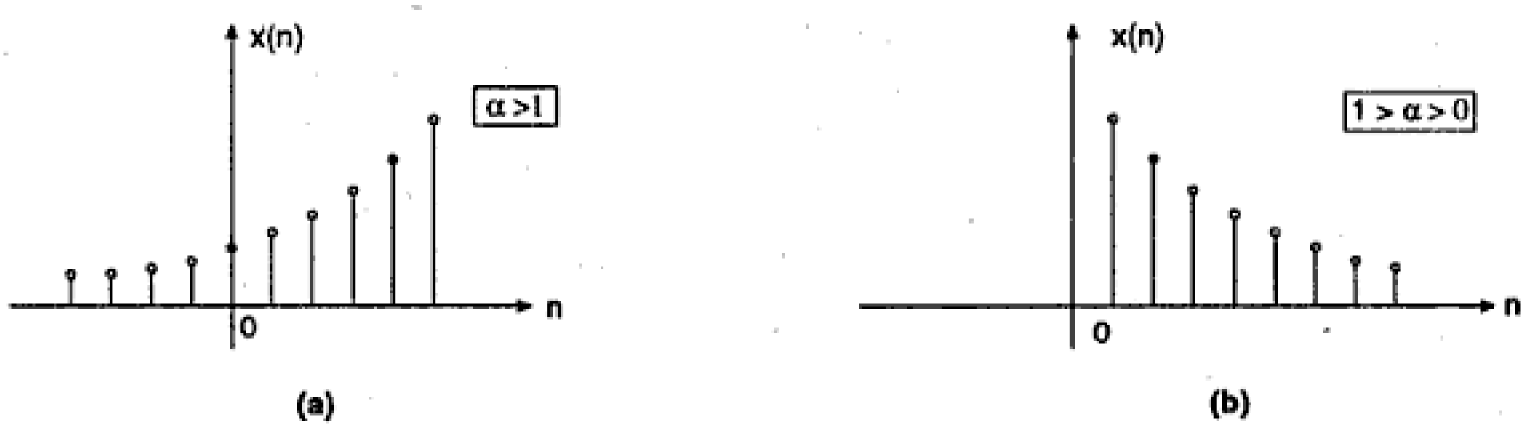
where B is scaling factor, (real parameter) and α is real parameter. Depending on the value of α, a real exponential signal can be further divided into four signals.

For $\alpha > 1$, the magnitude of a real exponential signal rises (increases) exponentially and is illustrated in Fig. 2.38(a).

For $1 > \alpha > 0$, the magnitude of a real exponential signal decays (decreases) exponentially and is illustrated in Fig. 2.38(b).

For $0 > \alpha > -1$, the magnitude of a real exponential signal decays (decreases) exponentially. For each integer value of n, the signal is represented alternatively and is illustrated in Fig. 2.38(c).

For $\alpha < -1$, the magnitude of a real exponential signal grows (increases) exponentially. For each integer value of n, the signal is represented alternatively and is illustrated in Fig. 2.38(d).



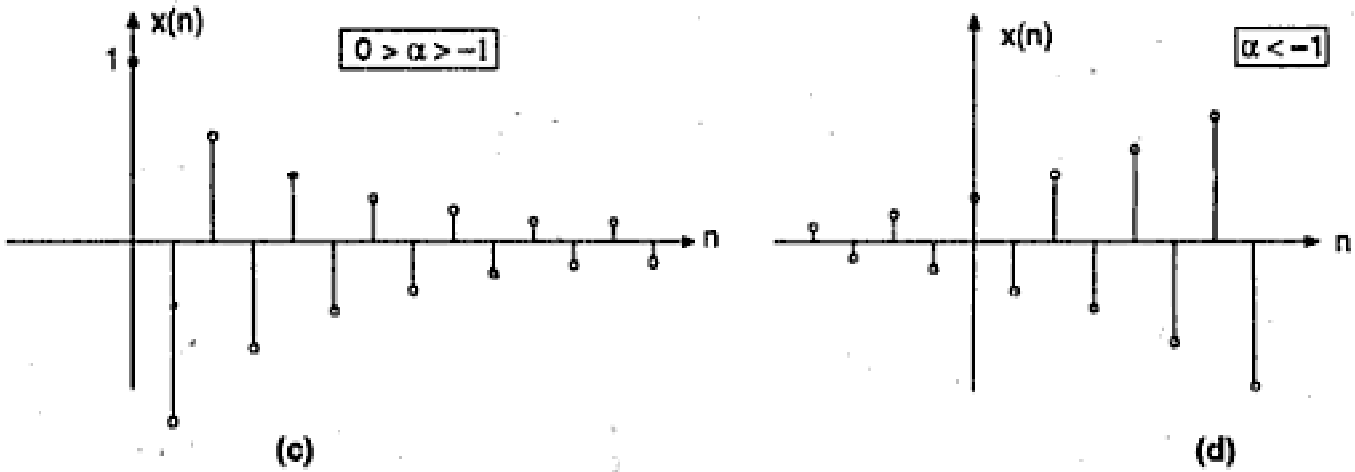


Fig. 2.38 Discrete-time Exponential Signal
(a) α > 1, (b) 1 > α > 0, (c) 0 > α > −1, (d) α < −1

Complex exponential signal (continuous-time) Let us define an exponential signal as

$$x(t) = e^{j\omega_0 t} \tag{2.50}$$

An important property of this complex exponential signal is its periodicity, i.e.

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

$$e^{j\omega_0 t} = e^{j\omega_0 t} e^{j\omega_0 T}$$

We know that,

$$e^{j\omega_0 T} = \cos(\omega_0 T) + j\sin(\omega_0 T) = 1$$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T.

If $\omega_0 \neq 0$, then the fundamental frequency ω_0 of the signal is the smallest positive frequency for which the equation $\omega_0 = 2\pi/T_0$ holds good.

Complex exponential signal (discrete-time) Let us define an exponential signal as

$$x(n) = C\alpha^n \tag{2.51}$$

where C = scaling factor, complex parameter
$\alpha = e^{\beta}$, complex parameter

2.4.2 Sinusoidal Signal (Continuous-time)

The continuous-time version of a sinusoidal signal $x(t)$ in its general form may be written as

$$X(t) = A\cos(\omega t + \phi) \tag{2.52}$$

where A = Amplitude of the signal $x(t)$
ω = Frequency of the signal $x(t)$ (radian/s)
ϕ = Phase angle of the signal $x(t)$ (radians)

and is illustrated in Fig. 2.39.

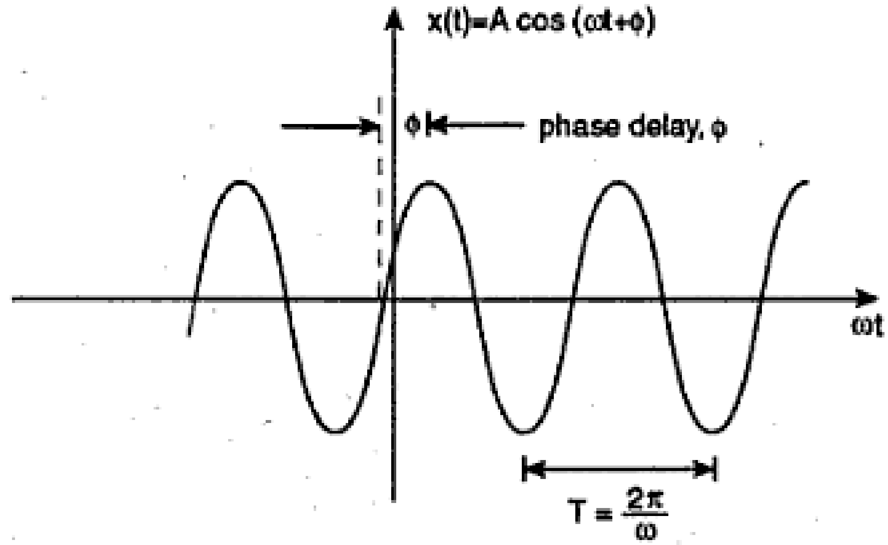


Fig. 2.39 Continuous-time Sinusoidal Signal

A sinusoidal signal is an example of a periodic signal, whose period $T = \frac{2\pi}{\omega}$.

In order to prove the periodicity of the sinusoidal signal, let us consider a continuous-time signal

$$x(t) = A\cos(\omega t + \phi) \tag{2.53}$$

which repeats every T second, i.e.

$$x(t+T) = A\cos(\omega(t+T) + \phi)$$
$$x(t+T) = A\cos(\omega t + \omega T + \phi)$$

For $T = \frac{2\pi}{\omega}$ (one cycle)

Then, the equation reduces to

$$x(t+T) = A\cos(\omega t + \phi) = x(t) \tag{2.54}$$

Similarly, let us consider a discrete-time signal

$$x(n) = \cos(\Omega n + \phi) \tag{2.55}$$

which repeats every T second, i.e.

$$x(n+N) = A\cos(\Omega(n+N) + \phi)$$
$$x(n+N) = A\cos(\Omega n + \Omega N + \phi)$$

where Ω = Angular frequency for a discrete-time signal

For $N = \frac{2\pi}{\Omega} m$ (one cycle)

$$x(n+N) = A\cos(\Omega n + 2\pi m + \phi)$$

Then, the equation reduces to

$$x(n+N) = A\cos(\Omega n + \phi) = x(n) \tag{2.56}$$

Complex exponential representation of sinusoidal signal The complex exponential signal can be expressed in terms of sinusoids as

$$e^{j\omega T} = \cos(\omega t) + j\sin(\omega t) \tag{2.57}$$

$$e^{-j\omega T} = \cos(\omega t) - j\sin(\omega t) \tag{2.58}$$

$$A\cos(\omega t + \phi) = \frac{A}{2}\left(e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right)$$

$$A\cos(\omega t + \phi) = \frac{A}{2}\left(e^{j\omega t}e^{j\phi} + e^{-j\omega t}e^{-j\phi}\right)$$

$$A\cos(\omega t + \phi) = \mathrm{Re}\{Ae^{j(\omega t+\phi)}\} \tag{2.59}$$

$$A\sin(\omega t + \phi) = \mathrm{Im}\{Ae^{j(\omega t+\phi)}\} \tag{2.60}$$

Similarly, for discrete-time signals, the complex exponential signal can be expressed in terms of sinusoids as

$$e^{j\Omega n} = \cos(\Omega n) + j\sin(\Omega n) \tag{2.61}$$

$$e^{-j\Omega n} = \cos(\Omega n) - j\sin(\Omega n) \tag{2.62}$$

Similarly, the discrete-time sinusoidal signal can be written in terms of periodic complex exponentials, again with the same fundamental period, Ω

$$A\cos(\Omega n + \phi) = \frac{A}{2}\left(e^{j(\Omega n+\phi)} + e^{-j(\Omega n+\phi)}\right)$$

$$A\cos(\Omega n + \phi) = \frac{A}{2}\left(e^{j\Omega n}e^{j\phi} + e^{-j\Omega n}e^{-j\phi}\right)$$

$$A\cos(\Omega n + \phi) = \mathrm{Re}\{Ae^{j(\Omega n+\phi)}\} \tag{2.63}$$

$$A\sin(\Omega n + \phi) = \mathrm{Im}\{Ae^{j(\Omega n+\phi)}\} \tag{2.64}$$

Exponentially damped sinusoidal signal When a real value decayed exponential signal is multiplied with a sinusoidal signal, it results in an exponentially damped sinusoidal signal. This can be represented by

$$x(t) = Ae^{-\alpha t}\sin(\omega t + \phi) \quad \alpha > 0 \tag{2.65}$$

and is illustrated in Fig. 2.40 (a).

Similarly, for discrete-time signal, an exponentially damped sinusoidal signal can be represented by

$$x(n) = Ae^{-\alpha n}\sin(\Omega n + \phi) \qquad \alpha > 0 \tag{2.66}$$

and is illustrated in Fig. 2.40 (b).

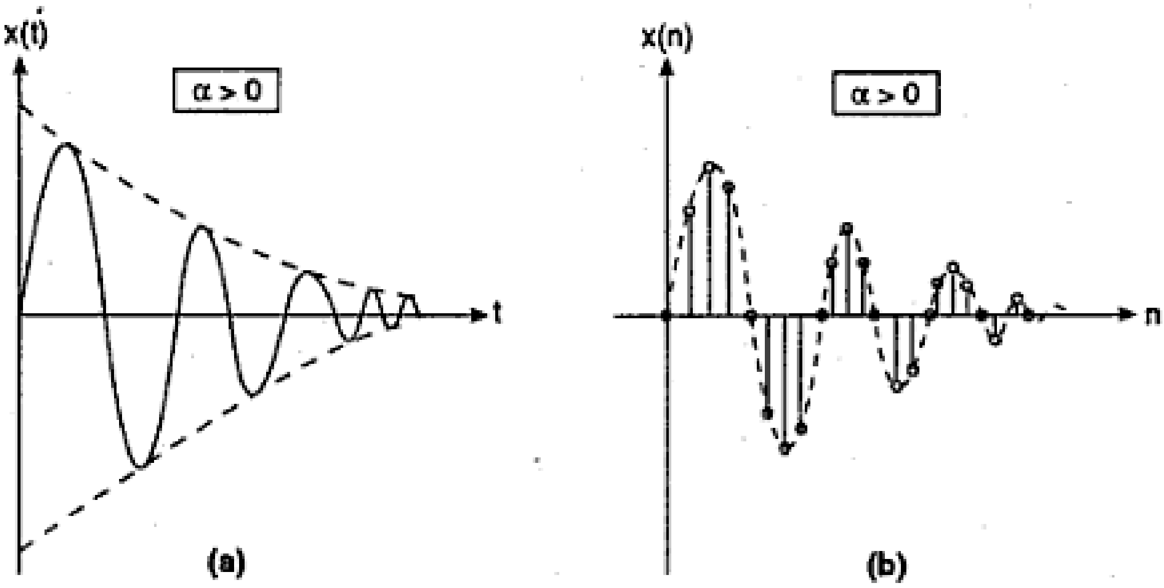


Fig. 2.40 Exponentially Damped Sinusoidal Signal
(a) Continuous-time Signal, (b) Discrete-time Signal

2.4.3 Step Function

The continuous-time step function is commonly denoted by $u(t)$ and is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.67}$$

and is illustrated in Fig. 2.41 (a).

The discrete-time step function is commonly denoted by $u(n)$ and is defined as

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{2.68}$$

and is illustrated in Fig. 2.41(b).

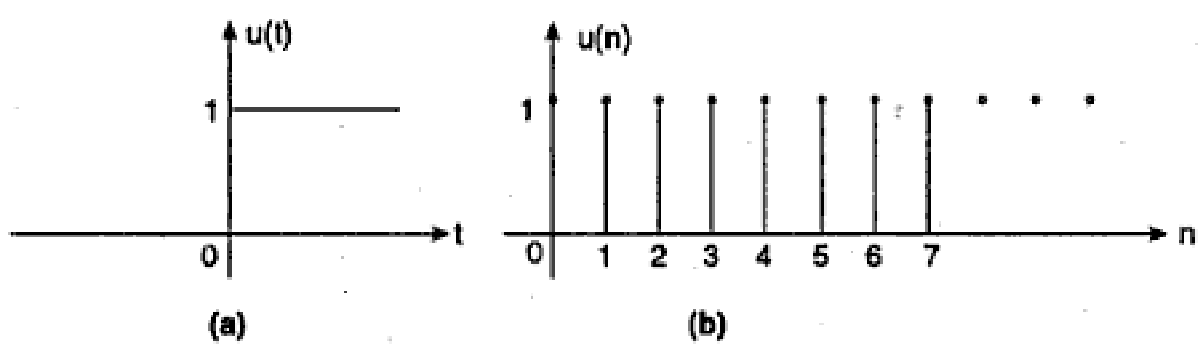


Fig. 2.41 Step Signal
(a) Continuous-time Signal (b) Discrete-time Signal

The step function shows discontinuity at $t = 0$ ($n = 0$ for discrete-time) in case of continuous-time step representation.

2.4.4 Impulse Function

The impulse function is a derivative of the step function $u(t)$ with respect to time. Conversely, the step function $u(t)$ is the integral of the impulse with respect to time. The continuous-time unit impulse function is commonly denoted by $\delta(t)$ and is defined as

$$\delta(t) = 0, \quad t \neq 0 \tag{2.69}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{2.70}$$

Equation (2.69) says that the impulse $\delta(t)$ is zero everywhere except at the origin, and equation (2.70) says that the area under the unit impulse is unity. The impulse function is also known as the Dirac delta function and is illustrated in Fig. 2.42(a).

The discrete-time impulse function is commonly denoted by $\delta(n)$ and is defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{2.71}$$

From equation (2.71), the impulse $\delta(n)$ is zero everywhere except at the origin, whose magnitude is unity. The impulse function is illustrated in Fig. 2.42(b).

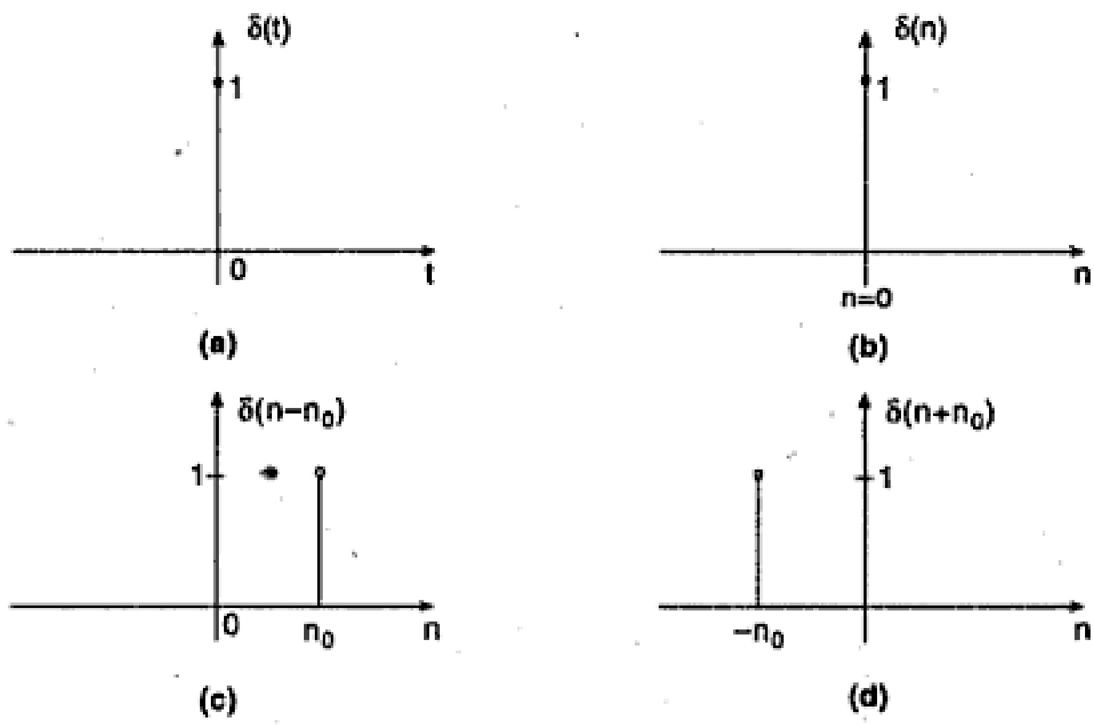


Fig. 2.42 Impulse Signal
(a) Continuous-time Impulse (b) Discrete-time Impulse
(c) Discrete-time Input Shift by $+n_0$ (d) Discrete-time Impulse Shift by $-n_0$

2.4.5 Ramp Function

The integral of the step function $u(t)$ is a ramp function of unit slope.

The continuous-time ramp function is commonly denoted by $r(t)$ and is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.72}$$

and is illustrated in Fig. 2.43(a).

The discrete-time ramp function is commonly denoted by $r(n)$ and is defined as

$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{2.73}$$

is illustrated in Fig. 2.43(b).

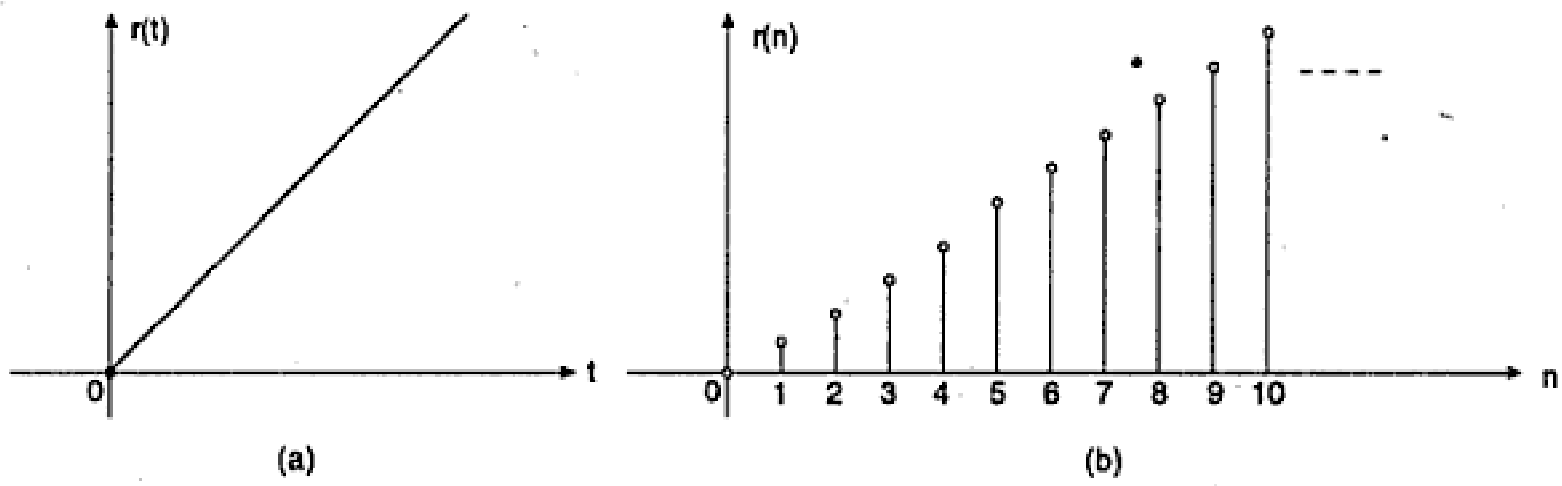


Fig. 2.43 ***Ramp Signal***
(a) Continuous-time Signal (b) Discrete-time Signal

■ 2.5 SYSTEM

A system is an entity that manipulates one or more input signals to perform a function, which results in a new output signal. A typical prototype of a system is shown in Fig. 2.44.

Fig. 2.44 ***Basic System***

In a communication transmitter, the system accepts message signal (audio, video or data), processes it (modulation and filtering), and gives an acceptable output for communication. A typical prototype of communication transmitter system is shown in Fig. 2.45.

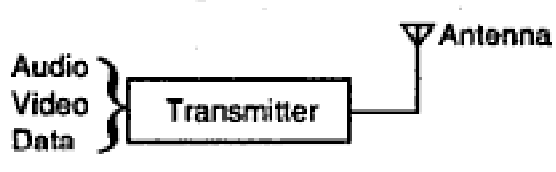


Fig. 2.45 ***Communication Transmitter System***

An electrocardiograph is a system which collects electrical potential from the surface of the heart, processes and filters it, and gives an electrical output for diagnosis. A typical prototype is it shown in Fig. 2.46.

Fig. 2.46 ***Electrocardiograph***

Specifically, a system is an interacting group of physical objects or conditions called system components. The system accepts one or more input signals or parameters, and produces one or more output signals or quantities.

2.6 PROPERTIES OF SYSTEMS

The properties of a system help us to understand the characteristic of the operator (say *H*) operating/representing the system. Following are a few basic properties of systems. Based on the properties, the system can be classified as

1. Continuous-time and Discrete-time system
2. Stable and Unstable system
3. Memory and Memoryless system
4. Invertible and Noninvertible system
5. Time-variant and Time-invariant system
6. Linear and Nonlinear system
7. Causal and Noncausal system

2.6.1 Continuous-Time and Discrete-Time System

Continuous-time system If the input and output of the system are continous-time signals, then the system is called 'Continuous-time system'.

Let us consider an input signal $x(t)$ to the system. If the system produces an output signal $y(t)$, then the system is called 'Continuous-time system'.

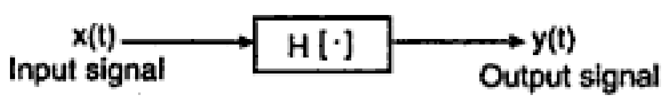


Fig. 2.47 Continuous-time System

Discrete-time system If the input and output of the system are discrete-time signals, then the system is called 'Discrete-time system'.

Let us consider an input signal $x(n)$ to the system. If the system produces an output signal $y(n)$, then the system is called 'Discrete-time system'.

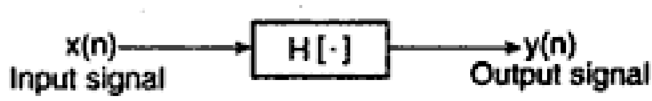


Fig. 2.48 Discrete-time System

2.6.2 Stable and Unstable System

Stable system A given system is said to be stable if and only if every bounded input produces a bounded output. The stable system is also known as 'Bounded Input–Bounded Output' (BIBO).

Let us consider a system operated by an operator H[·]. The system is said to be stable if the bounded input $x(t)$ produces bounded output $y(t)$, i.e.

if $$|x(t)| \leq M_x < \infty \text{ for all } t \tag{2.74a}$$

then $$|y(t)| \leq M_y < \infty \text{ for all } t \tag{2.74b}$$

A similar analysis holds good for a discrete-time system also, i.e.

if $$|x(n)| \le M_x < \infty \text{ for all } n \tag{2.75a}$$

then $$|y(n)| \le M_y < \infty \text{ for all } n \tag{2.75b}$$

A bounded input always has a finite value at infinity. The following are a few bounded signals.

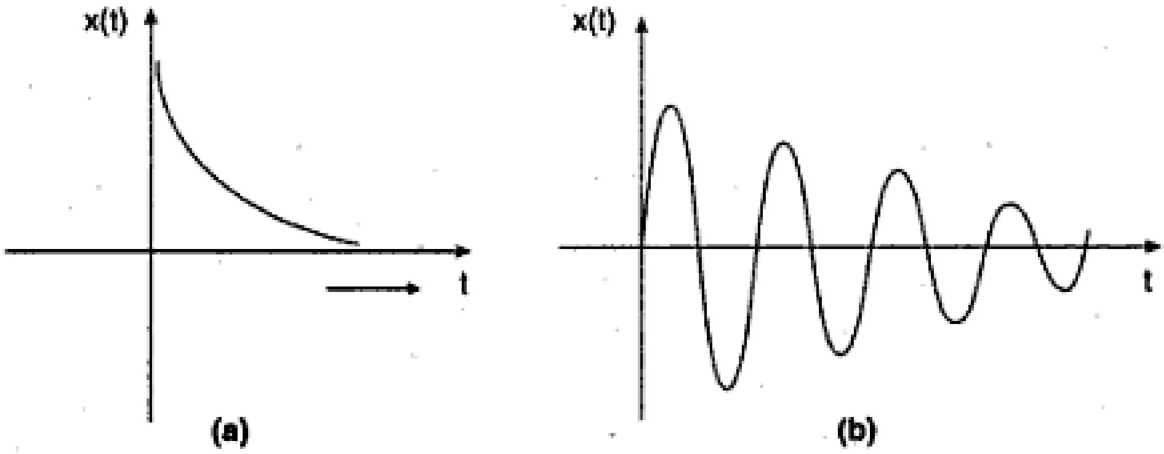


Fig. 2.49 Bounded Signals
(a) Decay Exponential (b) Sinusoidal Signal

Fig. 2.49(a) shows an exponential decay signal whose value is finite as $t \to \infty$. Similarly, Fig. 2.49(b) is a sinusoidal signal that maintains the finite magnitude as $t \to \infty$.

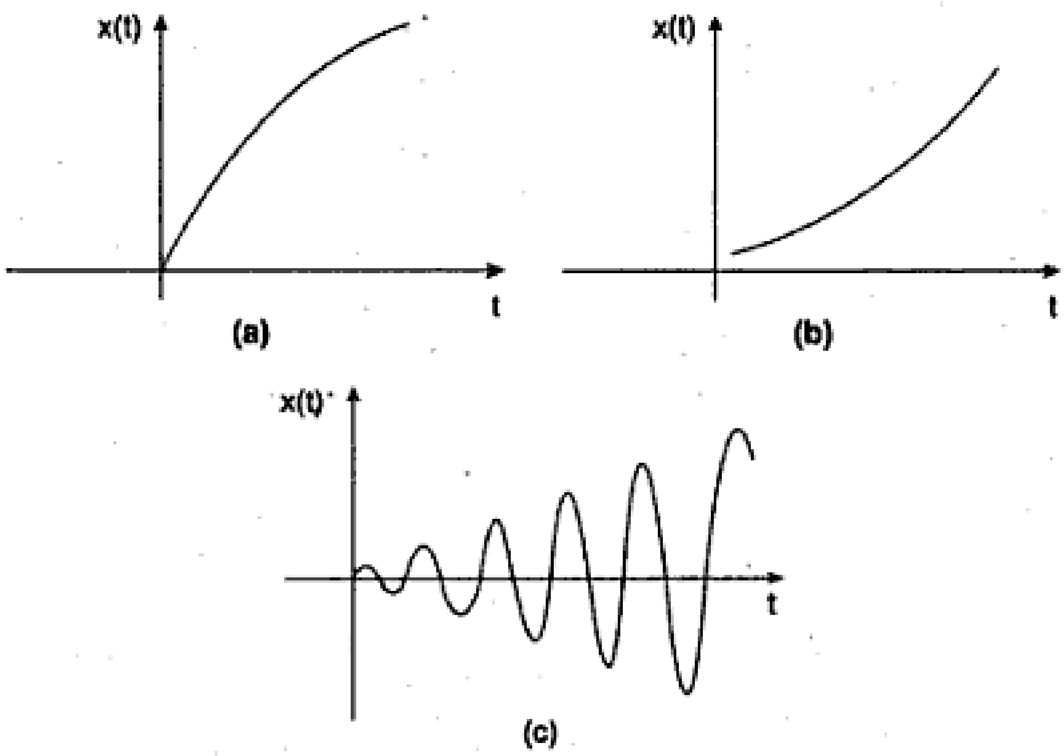


Fig. 2.50 Unbounded Signals

(a), (b) Rising Exponential
(c) Sinusoidal Signal with Exponential Rising Magnitude

Fig. 2.50 shows the various types of unbounded signals. Figs. 2.50(a) and (b) are rising exponentials whose values are infinite as $t \to \infty$. Most of the unbounded signals exhibit divergent property. Fig. 2.50(c) too is an unbounded signal as the magnitude of the sinusoid increases as $t \to \infty$.

SOLVED PROBLEMS

Problem 2.37 Determine whether the given system $h(n) = a^n\, u(n)$ is stable or not.

Solution For stability, the response of the system $h(n)$ must be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} \left| a^n u(n) \right| < \infty$$

$$\sum_{n=0}^{\infty} \left| a^n \right| = \frac{1}{1-a} < \infty, \text{ if } |a| < 1$$

There, the given system is unstable if and only if $|a| < 1$.

Problem 2.38 Determine whether the given system $h(n) = e^{an} u(n)$ is stable or not.

Solution For stability, the system response $h(n)$ must be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} \left| e^{an} u(n) \right| < \infty$$

$$\sum_{n=0}^{\infty} \left| e^{an} \right| = \frac{1}{1-a} < \infty, \text{ if } |a| < 1$$

Therefore, the given system is unstable.

Problem 2.39 Test whether the following discrete-time systems are stable or not.

(i) $h_1(n) = 2^n u(n-3)$ (ii) $h_2(n) = e^{n/2} u(n-4)$

(iii) $h_3(n) = e^{-2|n|}$ (iv) $h_4(n) = n u(n)$

(v) $h_5(n) = 3^n u(-n)$ (vi) $h_6(n) = 0.2^n u(n-3)$

Solution The general condition for stability is $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$.

(i) $h_1(n) = 2^n u(n-3)$

The condition for stability is $\sum_{n=3}^{\infty} \left| 2^n \right| < \infty$

$$\sum_{n=3}^{\infty} \left| 2^n \right| = 2^3 + 2^4 + 2^5 + \ldots + 2^{\infty} = \infty$$

Hence, the system is stable.

(ii) $h_2(n) = e^{n/2}u(n-4)$

The condition for stability is $\sum_{n=4}^{\infty}\left| e^{n/2}\right| < \infty$

$$\sum_{n=4}^{\infty}(e^{1/2})^n = e^2 + e^{5/2} + e^3 + e^{7/2} + e^4 + \ldots + e^{\infty} = \infty$$

Hence, the system is stable.

(iii) $h_3(n) = e^{-2|n|}$

The condition of stability is $\sum_{n=-\infty}^{\infty}\left|e^{-2|n|}\right| < \infty$

$$\sum_{n=-\infty}^{\infty} e^{-2|n|} = \sum_{n=-\infty}^{-1} e^{-2(-n)} + \sum_{n=0}^{\infty} e^{-2n}$$

$$\sum_{n=-\infty}^{\infty} e^{-2|n|} = \sum_{n=-\infty}^{-1} e^{2n} + \sum_{n=0}^{\infty} e^{-2n}$$

$$\sum_{n=-\infty}^{\infty} e^{-2|n|} = \sum_{n=1}^{\infty} e^{-2n} + \sum_{n=0}^{\infty} e^{-2n}$$

$$\sum_{n=-\infty}^{\infty} e^{-2|n|} = \frac{e^{-2}}{1-e^{-2}} + \frac{1}{1-e^{-2}} = \frac{1+e^{-2}}{1-e^{-2}} < \infty$$

Hence, the system is stable.

Hint $\sum_{n=k}^{\infty}\beta^n = \frac{\beta^k}{1-\beta}, |\beta| < 1$

(iv) $h_4(n) = nu(n)$

The condition for stability is $\sum_{n=0}^{\infty}|n| < \infty$

$$\sum_{n=0}^{\infty} n = 0 + 1 + 2 + 3 + \ldots + = \infty$$

Hence, the system is stable.

(v) $h_5(n) = 3^n u(-n)$

The condition for stability is $\sum_{n=-\infty}^{-1}\left|3^n\right| < \infty$

$$\sum_{n=-\infty}^{-1} 3^n = \sum_{n=1}^{\infty} 3^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$$

$$\sum_{n=-\infty}^{-1} 3^n = \frac{1/3}{1-(1/3)} = \frac{1}{2} < \infty$$

Hence, the system is stable.

Hint $\sum_{n=k}^{\infty}\beta^n = \frac{\beta^k}{1-\beta}, |\beta| < 1$

(vi) $h_6(n) = 0.2^n u(n-3)$

The condition for stability is $\sum_{n=3}^{\infty}(0.2)^n < \infty$

$$\sum_{n=3}^{\infty}(0.2)^n < \infty = \frac{(0.2)^3}{1-0.2} = 0.01 < \infty$$

Hence, the system is stable.

Problem 2.40 Test whether the following continuous-time systems are stable or not.

(i) $h_1(t) = e^{-a|t|}$ (ii) $h_2(t) = e^{2t}u(t+4)$

(iii) $h_3(t) = e^{-4t}u(t-4)$ (iv) $h_4(t) = te^{-at}u(t)$

(v) $h_5(t) = te^{+at}u(t)$ (vi) $h_8(t) = e^{-at}\cos bt\, u(t)$

(vii) $h_9(t) = e^{-at}\sin bt\, u(t)$

Solution The general condition for stability is $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$.

(i) $h_1(t) = e^{-a|t|}$

The condition for stability is $\int_{-\infty}^{\infty} e^{-a|t|}dt < \infty$

$$\int_{-\infty}^{\infty} e^{-a|t|}dt = \int_{-\infty}^{0} e^{-a(-t)}dt + \int_{0}^{\infty} e^{-a(t)}dt$$

$$\int_{-\infty}^{\infty} e^{-a|t|}dt = \frac{1}{a}e^{at}\Big|_{-\infty}^{0} + \frac{1}{-a}e^{-at}\Big|_{0}^{\infty}$$

$$\int_{-\infty}^{\infty} e^{-a|t|}dt = \frac{1}{a}\left[e^0 - e^{-\infty}\right] - \frac{1}{a}\left[e^{-\infty} - e^0\right]$$

$$\int_{-\infty}^{\infty} e^{-a|t|}dt = \frac{1}{a}[1-0] - \frac{1}{a}[0-1]$$

$$\int_{-\infty}^{\infty} e^{-a|t|}dt = \frac{1}{a} + \frac{1}{a} = \frac{2}{a} < \infty$$

Hence, the system is stable $(a \neq 0)$.

(ii) $h_2(t) = e^{2t}u(t+4)$

The condition for stability is $\int_{-4}^{\infty} \left|e^{2t}\right| dt < \infty$

$$\int_{-4}^{\infty} e^{2t}dt = \frac{1}{2}e^{2t}\Big|_{-4}^{\infty}$$

$$\int_{-4}^{\infty} e^{2t}dt = \frac{1}{2}(e^{\infty} - e^{-8}) = \infty$$

Hint Since $e^{+\infty} = \infty$

Hence, the system is stable.

(iii) $h_3(t) = e^{-4t}u(t-4)$

The condition for stability is $\int_4^{\infty}\left| e^{-4t}\right| dt < \infty$

$$\int_4^{\infty} e^{-4t}dt = \frac{-1}{4}e^{-4t}\Big|_4^{\infty}$$

$$\int_4^{\infty} e^{-4t}dt = -\frac{1}{4}\left[e^{-\infty} - e^{-16}\right]$$

$$\int_4^{\infty} e^{-4t}dt = -\frac{1}{4}\left[0 - e^{-16}\right] = \frac{e^{-16}}{4} < \infty$$

Hence, the system is stable.

(iv) $h_4(t) = te^{-at}u(t)$

The condition for stability is $\int_0^{\infty}\left| te^{-at}\right| dt < \infty$

Applying Bernouli's theorem

$$\int_0^{\infty} te^{-at}dt = t.\frac{e^{-at}}{-a}\Big|_0^{\infty} - \frac{e^{-at}}{a^2}\Big|_0^{\infty}$$

$$= [0-0] - \left[0 - \frac{1}{a^2}\right]$$

$$\int_0^{\infty} te^{-at}dt = \frac{1}{a^2} < \infty$$

Hint $\int u.v\,dt = uv_1 - u'v + u''v_3 \ldots\ldots$

$v_1 = \int v\,dt, v_2 = \int v_1\,dt$

Hence, the system is stable.

(v) $h_5(t) = te^{+at}u(t)$

The condition for stability is $\int_0^{\infty}\left| te^{at}\right| dt < \infty$

Applying Bernouli's theorem

$$\int_0^{\infty} te^{at}dt = t.\frac{e^{at}}{a}\Big|_0^{\infty} - \frac{e^{at}}{a^2}\Big|_0^{\infty}$$

Hint $\int u.v\,dt = uv_1 - u'v + u''v_3 \ldots\ldots$

$v_1 = \int v\,dt . v_2 = \int v_1\,dt$

$$\int_0^{\infty} te^{at}dt = \infty$$

Hence, the system is stable.

(vi) $h_8(t) = e^{-at}\cos bt\, u(t)$

The condition for stability is $\int_0^\infty \left| e^{-at} \cos bt \right| dt < \infty$

Hint $\int_a^b e^{gx}\cos cx\, dx = \dfrac{e^{gx}}{g^2+c^2}[g\cos(cx) + c\sin(cx)]\Big|_a^b$

$$\int_0^\infty e^{-at}\cos bt\, dt = \left\{\frac{e^{-at}}{a^2+b^2}[-a\cos bt + b\sin bt]\right\}_0^\infty$$

$$\int_0^\infty e^{-at}\cos bt\, dt = \frac{1}{a^2+b^2}\{ e^{-\infty}[-a\cos(\infty) + b\sin(\infty)]$$

$$-e^0[-a\cos(0) + b\sin(0)]\}$$

$$\int_0^\infty e^{-at}\cos bt\, dt = \frac{1}{a^2+b^2}\ (a)$$

$$\int_0^\infty e^{-at}\cos bt\, dt = \frac{a}{a^2+b^2} < \infty$$

Hence, the system is stable.

(vii) $h_9(t) = e^{-at}\sin bt\, u(t)$

The condition for stability is,

$$\int_0^\infty e^{-at}\sin bt\, dt = \left\{\frac{e^{-at}}{a^2+b^2}[-a\sin bt - b\cos bt]_0^\infty\right\}$$

Hint $\int_0^\infty e^{gx}\sin cx\, dx = \dfrac{e^{gx}}{g^2+c^2}[g\sin(cx) - c\cos cx]\Big|_a^b$

$$\int_0^\infty e^{-at}\sin bt\, dt = \frac{1}{a^2+b^2}\left[e^{-\infty}(-a\sin\infty - b\cos\infty)\right]$$

$$-\left[e^0(-a\sin 0 - b\cos 0)\right]$$

$$\int_0^\infty e^{-at}\sin bt\, dt = \frac{1}{a^2+b^2}(b) = \frac{b}{a^2+b^2} < \infty$$

Hence, the system is stable.

2.6.3 Memory and Memoryless System

Memory system The given system is said to possess memory if the output of the system depends on past and future values.

Examples

$$y(t) = x(t) + x(t-1) + x(t+1)$$

$$y(t) = \frac{dx(t)}{dt}$$

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n+1)]$$

$$y(n) = x(n-1) = x(n+1)$$

The memory system is also known as a 'Dynamic System'. An inductor has memory since the charge is stored as current. The current $i(t)$ flowing through it is related to the applied voltage $v(t)$, i.e.

$$v(t) = L\frac{di(t)}{dt}$$

where L = Inductance of the inductor

Similarly, capacitor is a memory element, which stores the charge as voltage. The voltage across a capacitor is related to the current flowing through it, i.e.

$$i(t) = C\frac{dv(t)}{dt}$$

where C = Capacitance of the capacitor

Note The analogous to differential equation in a continuous-time system is a difference equation in the discrete-time system.

$$\frac{dy(t)}{dt} \equiv y(n-1) - y(n)$$

Therefore, the differential equation is a memory system.

Memoryless system The given system is said to be memoryless if the output of the system depends solely on the present value.

Examples

$$y(t) = x(t)$$

$$y(t) = x^2(t)$$

$$y(n) = x(n)$$

$$y(n) = nx(n)$$

The memoryless system is also known as a 'Static System'. A resistor is a memoryless system since the current flow $i(t)$ flowing through it is proportional to the applied potential, i.e.

$$i(t) = \frac{v(t)}{R}$$

where R = Resistance of the resistor

2.6.4 Invertible and Noninvertible System

Invertible system A system is said to be an invertible system if the input signal given to the system can be recovered.

The concept of invertibility is illustrated in Fig. 2.51.

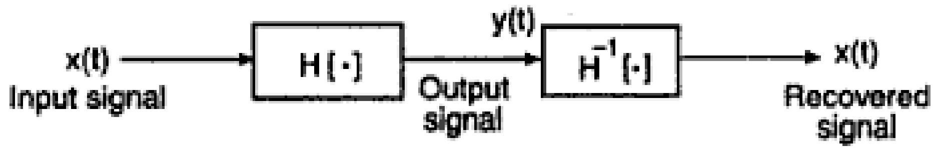


Fig. 2.51 Invertible System

Let us consider an example of a communication system shown in Fig. 2.52,where the signal is modulated with a carrier signal and transmitted through the transmitter system. The same signal is recovered at the receiver system by demodulating the signal from the carrier signal.

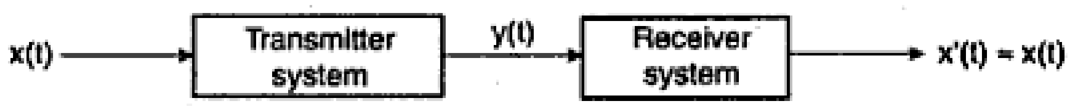


Fig. 2.52 Communication System

All the transforms used in signal processing are invertible system.

Noninvertible system A system is said to be noninvertible if the input signal given to the system cannot be recovered from the output signal of the system.

The square-law system is generally a noninvertible system, except for the distinct inputs.

2.6.5 Time-invariant and Time-variant System

Time-invariant system A system is said to be time-invariant if the input signal is delayed or advanced by any factor that leads to some delay or advancement in the time scale by the same factor, i.e. the system responds to an input which is given at any instant of time and results in an output.

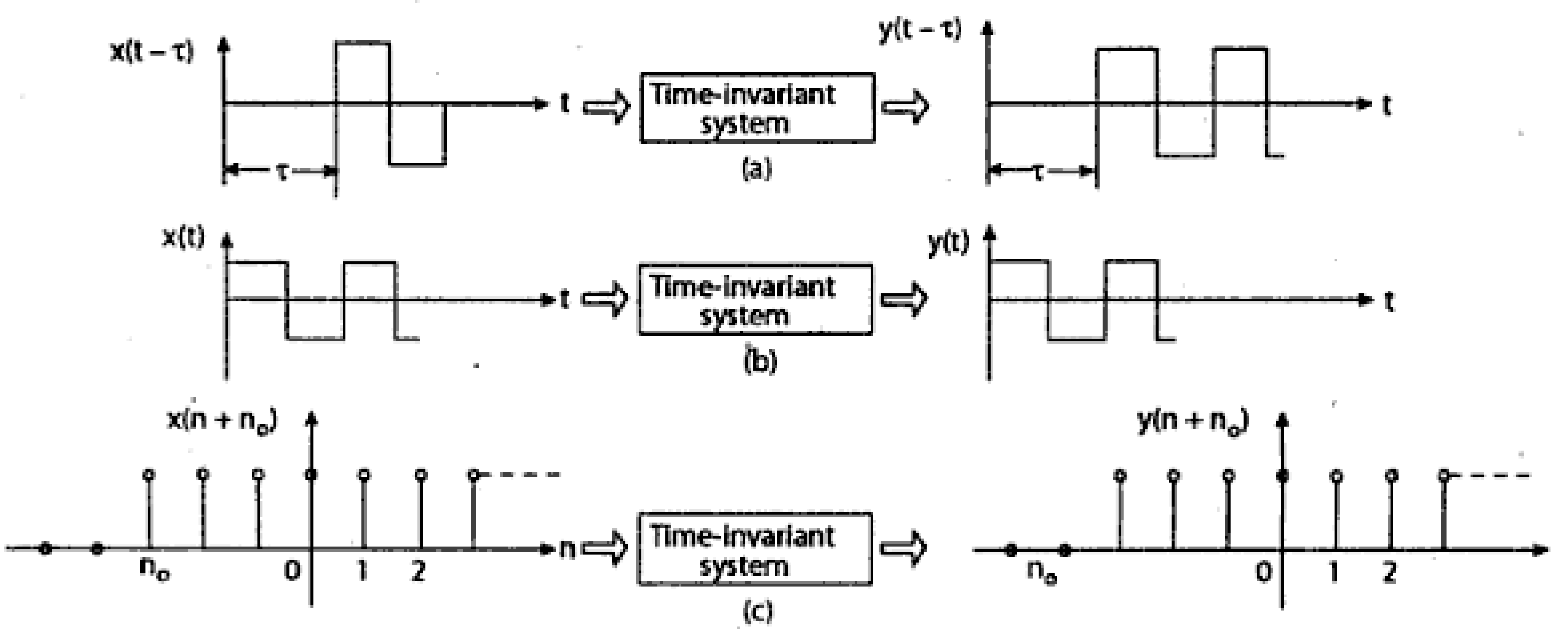


Fig. 2.53 Basic Time-invariant Systems
(a) Delayed Time-invariant System (b) Time-invariant System
(c) Advanced Time-invariant System

Let us consider three time-invariant situation depicted in Fig. 2.53. Fig. 2.53(a) illustrates a delayed pulse output (τ sec) produced by a system for a delayed pulse input of τ sec. Fig. 2.53(b) illustrates a pulse output produced by a system for a pulse input where in delay is not incurred neither in input signal. Fig. 2.53 (c) illustrates an advanced discrete step output produced by system for an advanced discrete step input. Any other combination (input and output combination) of the system is time-variant system.

SOLVED PROBLEM

Problem 2.41 The input-output relation is given by $y(t) = \sin[x(t)]$. Determine whether the system is time-invariant or not.

Solution

$$y(t) = \sin[x(t)]$$

Let us assume the signal of the form

$$y_1(t) = \sin[x_1(t)] \tag{1}$$

Let us introduce time delay t_0 in the input signal in equation (1), then

$$x_2(t) = x_1(t - t_0)$$

The delay input therefore results in the output

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)] \tag{2}$$

Let us introduce the same time delay t_0 in the output of the equation, i.e.

$$y_1(t - t_0) = \sin[x_1(t - t_0)] \tag{3}$$

On comparing equations (2) and (3),

$$y_2(t) = y_1(t - t_0)$$

Hence, system is time-invariant.

Time-variant system A system is said to be time-variant if the output signal is delayed or advanced with respect to input signal as shown in Fig. 2.54.

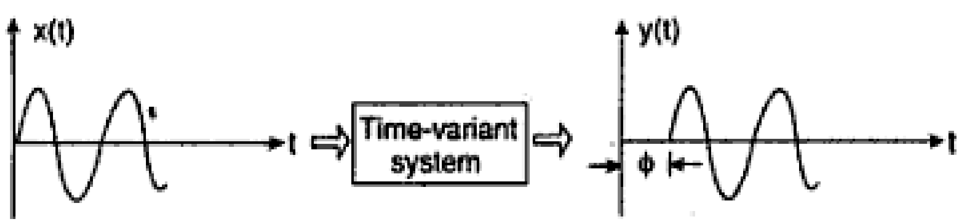


Fig. 2.54 Time-variant System

SOLVED PROBLEMS

Problem 2.42 The input-output relation is given by $y(t) = t\,x(t)$. Determine whether the system is time-variant or not.

Solution

$$y(t) = t\,x(t)$$

Let us assume the signal of the form

$$y_1(t) = t\,x_1(t) \tag{1}$$

Let us introduce time delay t_0 in the input signal in equation (1)

$$x_2(t) = x_1(t - t_0)$$

The delay in input results in an output

$$y_2(t) = t\,x_2(t) = t\,x_1(t - t_0) \tag{2}$$

Let us introduce the same delay t_0 in the output of the system

$$y(t-t_0)=(t-t_0)x_1(t-t_0) \tag{3}$$

On comparing equation (2) and (3),

$$y_2(t) \neq y_1(t - t_0)$$

Hence, the system is time-variant.

Problem 2.43 Determine whether the following systems are time-variant or not.

(i) $y(t) = x(t)\sin\omega t$ (ii) $y(t) = x(4t)$

(iii) $y(t) = e^{x(t)}$ (iv) $y(t) = t^2 x(t)$

Solution

(i) $y(t) = x(t)\sin\omega t$ (1)

$$y(t) = T[x(t)] = \sin\omega t\; x(t)$$

Introduce time delay t_0 in the input, i.e.

$$x_1(t) = x(t - t_0)$$

$$y_1(t) = \sin\omega t\; x_1(t) = \sin\omega t\; x(t - t_0) \tag{2}$$

Introduce time delay t_0 in the output of the equation, i.e.

$$y(t - t_0) = \sin\omega t\,(t - t_0)\,x(t - t_0) \tag{3}$$

On comparing equations (2) and (3),

$$y_1(t) \neq y(t - t_0)$$

The system is time-variant.

(ii) $y(t) = x(4t)$ (4)

Introduce time delay t_0 in the input, i.e.

$$x_1(4t) = x_1[4(t - t_0)]$$

$$y_1(t) = x_1[4(t - t_0)] \tag{5}$$

Introduce time delay t_0 in the output of the equation, i.e.

$$y(t - t_0) = x[4(t - t_0)] \tag{6}$$

On comparing equations (5) and (6),

$$y_1(t) = y(t - t_0)$$

The system is time-invariant.

(iii) $y(t) = e^{x(t)}$ (7)

$$y(t) = T[x(t)] = e^{x(t)}$$

Introduce time delay t_0 in the input, i.e.

$$x_1(t) = x(t - t_0)$$

$$y_1(t) = e^{x_1(t)} = e^{x(t - t_0)} \tag{8}$$

Introduce time delay t_0 in the output of the equation, i.e.

$$y(t - t_0) = e^{x(t - t_0)}. \tag{9}$$

On comparing equations (8) and (9),

$$y_1(t) = y(t - t_0)$$

The system is time-invariant.

(iv) $y(t) = t^2 x(t)$ (10)

$$y(t) = T[x(t)] = t^2 x(t)$$

Introduce time delay t_0 in the input, i.e.

$$x_1(t) = x(t - t_0)$$

then $$y_1(t) = t^2 x_1(t) = t^2 x(t - t_0) \tag{11}$$

Introduce time delay t_0 in the output, i.e.

$$y(t - t_0) = (t - t_0)^2 x(t - t_0) \tag{12}$$

On comparing equations (11) and (12),

$$y_1(t) \neq y(t - t_0)$$

The system is time-variant.

Problem 2.44 Detetmine whether the following systems described by the given differential equations are time-invariant or not.

(i) $\dfrac{d^2y(t)}{dt^2}+2\dfrac{dy(t)}{dt}+y(t)=x(t)$

(ii) $\dfrac{d^2y(t)}{dt^2}+2t\dfrac{dy(t)}{dt}+y(t)=x(t)$

(iii) $t^3\dfrac{d^3y(t)}{dt^3}+t^2\dfrac{d^2y(t)}{dt^2}+\dfrac{dy(t)}{dt}+y(t)=x(t)+\dfrac{dx(t)}{dt}$

Note *The differential equation indicates that the time-invariant system must have constant coefficients. If the coefficients are time-dependent parameter, then the system is called a time-variant system.*

Solution

(i) $\dfrac{d^2y(t)}{dt^2}+2\dfrac{dy(t)}{dt}+y(t)=x(t)$

Since the coefficients are constant multiples, the system described by this differential equation is a time-invariant system.

(ii) $\dfrac{d^2y(t)}{dt^2}+2t\dfrac{dy(t)}{dt}+y(t)=x(t)$

Since the coefficient of the second term is a time-dependent parameter, i.e. $2t$, the system described by the given differential equation is a time-variant system.

(iii) $t^3\dfrac{d^3y(t)}{dt^3}+t^2\dfrac{d^2y(t)}{dt^2}+t\dfrac{dy(t)}{dt}+y(t)=x(t)+\dfrac{dx(t)}{dt}$

Since the coefficients of the first three terms of the differential equation is time-dependent, the system described by the given differential equation is a time-variant system.

Problem 2.45 Determine whether the following discrete-time systems are time-invariant or not.

(i) $y(n)=\cos[x(n)]$ (ii) $y(n)=\ln[x(n)]$

(iii) $y(n)=x(n)-x(n-1)$ (iv) $y(n)=x(n)\,x(n+1)$

Solution

(i) $y(n)=\cos[x(n)]$

$$y(n)=T[x(n)]=\cos[x(n)] \tag{1}$$

Introduce time delay n_0 in the input, i.e.

$$x_1(n)=x(n-n_0)$$

then

$$y_1(n)=\cos[x_1(n)]=\cos[x(n-n_0)] \tag{2}$$

Introduce time delay n_0 in the output of the equation (1), i.e.

$$y(n-n_0)=\cos[x(n-n_0)] \tag{3}$$

On comparing equations (2) and (3),

$$y_1(n) = y(n-n_0)$$

The system is time-invariant.

(ii) $y(n) = \ln[x(n)]$ (4)

$$y(n) = T[x(n)] = \ln[x(n)]$$

Introduce time delay n_0 in the input, i.e.

$$x_1(n) = x(n-n_0) \tag{5}$$

then $$y_1(n) = \ln[\,x_1(n)] = \ln[x(n-n_0)] \tag{6}$$

Introduce time delay n_0 in the output of equation (4), i.e.

$$y(n-n_0) = \ln[x(n-n_0)]$$

On comparing equations (5) and (6),

$$y_1(n) = y(n-n_0)$$

The system is time-invariant.

(iii) $y(n) = x(n) - x(n-1)$ (7)

Introduce time delay n_0 in the input, i.e.

$$x_1(n) - x_1(n-1) = x(n-n_0) - x(n-1-n_0)$$

then $$y_1(n) = x_1(n) - x_1(n-1) = x(n-n_0) - x(n-1-n_0) \tag{8}$$

Introduce time delay n_0 in the output of equation (7), i.e.

$$y(n-n_0) = x(n-n_0) - x(n-1-n_0) \tag{9}$$

On comparing equations (8) and (9),

$$y_1(n) = y(n-n_0)$$

The system is time-invariant.

(iv) $y(n) = x(n)\,x(n+1)$ (10)

Introduce time delay n_0 in the input, i.e.

$$x_1(n)\,x_1(n+1) = x(n-n_0)\;x(n+1-n_0)$$

then $$y_1(n) = x_1(n)\;x_1(n+1) = x(n-n_0)\;x(n+1-n_0) \tag{11}$$

Introduce time delay n_0 in the output of equation (10), i.e.

$$y(n-n_0) = x(n-n_0)\,x(n+1-n_0) \tag{12}$$

On comparing equations (11) and (12),

$$y_1(n) = y(n-n_0)$$

The system is time-invariant.

Problem 2.46 Determine whether the following systems are time-invariant or not.

(i) $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$ [where C is capacitance of the capacitor]

(ii) $y(t) = \frac{x(t)}{R(t)}$ [where $R(t)$ is resistance of the thermistor]

Solution

(i) $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$ (1)

$$y(t) = T[x(\tau)] = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

Introduce time delay t_0 in the input, i.e.

$$x_1(\tau) = x(\tau - t_0)$$

$$y_1(t) = \frac{1}{C}\int_{-\infty}^{t} x_1(\tau) = \frac{1}{C}\int_{-\infty}^{t} x(\tau - t_0)\,d\tau \quad (2)$$

Introduce time delay t_0 in the output of equation (1), i.e.

$$y(t - t_0) = \frac{1}{C}\int_{-\infty}^{t-t_0} x(\tau - t_0)\,d\tau \quad (3)$$

On comparing equations (2) and (3),

$$y_1(t) = y(t - t_0)$$

The system is time-invariant.

(ii) $y(t) = \frac{x(t)}{R(t)}$

$$y(t) = T[x(t)] = \frac{x(t)}{R(t)} \quad (4)$$

Introduce time delay t_0 in the input, i.e.

$$x_1(t) = x(t - t_0)$$

then $$y_1(t) = \frac{x_1(t)}{R_1(t)} = \frac{x(t - t_0)}{R_1(t)} \quad (5)$$

Introduce time delay t_0 in the output of equation (4), i.e.

$$y(t - t_0) = \frac{x(t - t_0)}{R(t - t_0)} \quad (6)$$

On comparing equations (5) and (6),

$$y_1(t) \neq y(t-t_0)$$

The system is time-variant.

2.6.6 Linear and Nonlinear System

Let us consider an input $x_1(t)$ given to a continuous-time system which responds with $y_1(t)$. Similarly, let us consider another signal $x_2(t)$ given to the same continuous-time system which results in a response $y_2(t)$. Then the system is said to be linear if

1. The response to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$ (additive property)
2. The response to $ax_1(t)+bx_2(t)$ is $ay_1(t)+by_2(t)$, where a and b are complex constants (scaling property)

Mathematically,

$$T[ax_1(t)+bx_2(t)] = aT[x_1(t)]+bT[x_2(t)]$$
$$T[ax_1(t)+bx_2(t)] = ay_1(t)+by_2(t) \tag{2.76}$$

If an input consists of weighted sum of several signals, then the output is the weighted sum of the responses of the system to each of those signals.

Similarly, if two signals $x_1(n)$ and $x_2(n)$ given to a discrete-time system results in an output $y_1(n)$ and $y_2(n)$ respectively, then the system is said to be linear if

$$T[ax_1(n)+bx_2(n)] = aT[x_1(n)]+bT[x_2(n)]$$
$$T[ax_1(n)+bx_2(n)] = ay_1(n)+by_2(n) \tag{2.77}$$

The superposition property holds for a linear system in both continuous-time and discrete-time.

SOLVED PROBLEMS

Problem 2.47 Determine whether the given continuous-time system is linear or not.

$$y(t) = t\,x(t)$$

Solution Let us define the input signal $x_1(t)$ whose response $y_1(t)$ is given by

$$y_1(t) = t_1 x(t)$$

Similarly, let us define another signal $x_2(t)$ whose response $y_2(t)$ is

$$y_2(t) = t\,x_2(t)$$

The two defined signals are related by

$$x_3(t) = ax_1(t)+bx_2(t)$$

Then the output $y_3(t)$ is defined as

$$y_3(t) = t\,x_3(t)$$
$$y_3(t) = t\left[ax_1(t) + bx_2(t)\right]$$
$$y_3(t) = at\,x_1(t) + bt\,x_2(t)$$
$$y_3(t) = ay_1(t) + by_2(t)$$

Therefore, the system is linear.

Problem 2.48 Determine whether the given continuous-time system is linear or not.

$$y(t) = t\,x(t) + k$$

where k is a constant.

Solution Let us define the input signal $x_1(t)$ which results in an output $y_1(t)$.

$$y_1(t) = t\,x_1(t) + k$$

Similarly, let us define another signal $x_2(t)$ which results in an output $y_2(t)$.

$$y_2(t) = t\,x_2(t) + k$$

The above signals $x_1(t)$ and $x_2(t)$ are related as

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a, b = constants

Then, the output $y_3(t)$ is defined as

$$y_3(t) = t\,x_3(t) + k$$
$$y_3(t) = T\left[ax_1(t) + bx_2(t)\right] + k$$
$$y_3(t) = at\,x_1(t) + bt\,x_2(t) + k$$
$$y_3(t) \neq a\,y_1(t) + by_2(t)$$

Therefore, the system is nonlinear.

Problem 2.49 Determine whether the given discrete-time system is linear or not.

$$y(n) = n^2 x(n-1)$$

Solution Let us define the input signal $x_1(n-1)$ which results in an output $y_1(n)$.

$$y_1(n) = n^2 x_1(n-1)$$

Similarly, let us define another signal $x_2(n-1)$ which results in an output $y_2(n)$.

$$y_2(n) = n^2 x_2(n-1)$$

The above signals $x_1(n-1)$ and $x_2(n-1)$ are related as

$$x_3(n) = ax_1(n) + bx_2(n)$$

where a, b = constants

Then the output $y_3(n)$ is defined by

$$y_3(n) = n^2 x_3(n-1)$$
$$y_3(n) = n^2 [ax_1(n-1) + bx_2(n-1)]$$
$$y_3(n) = an^2 x_1(n-1) + bn^2 x_3(n-1)$$
$$y_3(n) = ay_1(n) + by_2(n)$$

Therefore, the system is linear.

Problem 2.50 Determine whether the given system is linear or not.

$$y(n) = x^2(n)$$

Solution Let us define the signal $x_1(n)$ whose response is given by

$$y_1(n) = x_1^2(n)$$

Similarly, let us define another signal $x_2(n)$ whose response is given by

$$y_2(n) = x_2^2(n)$$

These two signals are related as

$$x_3(n) = ax_1(n) + bx_2(n)$$

where a, b = constants

Then the output $y_3(n)$ becomes

$$y_3(n) = x_3^2(n)$$
$$y_3(n) = [ax_1(n) + bx_2(n)]^2$$
$$y_3(n) = a^2 x_1^2(n) + b^2 x_2^2(n) + 2x_1(n)x_2(n)$$
$$y_3(n) \neq ay_1(n) + by_2(n)$$

Therefore, the system is nonlinear.

Problem 2.51 Determine whether the following continuous-time systems are linear or not.

(i) $y(t) = x(\sin t)$ (ii) $y(t) = \sin[x(t)]$

(iii) $y(t) = t^2 x(t-1)$ (iv) $y(t) = e^{x(t)}$

Solution

(i) $y(t) = x(\sin t)$

Let us define $x_1(t)$ and $x_2(t)$ as

$$x_1(t) \Rightarrow y_1(t) = x_1[\sin(t)]$$
$$x_2(t) \Rightarrow y_2(t) = x_2[\sin(t)]$$

Let us define $x_3(t)$ such that

$$x_3(t) = ax_1(t) + bx_2(t)$$

then the output becomes,

$$y_3(t) = x_3[\sin(t)]$$
$$y_3(t) = ax_1[\sin(t)] + bx_2[\sin(t)]$$
$$y_3(t) = ay_1(t) + by_2(t)$$

Therefore, the system is linear.

(ii) $y(t) = \sin[x(t)]$

Let us define $x_1(t)$ and $x_2(t)$ as

$$x_1(t) \Rightarrow y_1(t) = \sin[x_1(t)]$$
$$x_2(t) \Rightarrow y_2(t) = \sin[x_2(t)]$$

Let us define $x_3(t)$ such that

$$x_3(t) = ax_1(t) + bx_2(t)$$

then the output becomes,

$$y_3(t) = \sin[x_3(t)] = \sin[ax_1(t) + bx_2(t)]$$

Hint $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$y_3(t) = \sin[ax_1(t)]\cos[bx_2(t)] + \cos[ax_1(t)]\sin[bx_2(t)]$$
$$y_3(t) \neq ay_1(t) + by_2(t)$$

Therefore, the system is nonlinear.

(iii) $y(t) = t^2 x(t-1)$

Let us define $x_1(t)$ and $x_2(t)$ as

$$x_1(t) \Rightarrow y_1(t) = t^2 x_1(t-1)$$
$$x_2(t) \Rightarrow y_2(t) = t^2 x_2(t-1)$$

Let us define $x_3(t)$ such that

$$x_3(t) = ax_1(t) + bx_2(t)$$

then the output becomes,

$$y_3(t) = t^2 x_3(t-1)$$
$$y_3(t) = t^2[ax_1(t-1) + bx_2(t-1)]$$
$$y_3(t) = at^2 x_1(t-1) + bt^2 x_2(t-1)$$
$$y_3(t) = ay_1(t) + by_2(t)$$

Therefore, the system is linear.

(iv) $y(t) = e^{x(t)}$

Let us define $x_1(t)$ and $x_2(t)$ as

$$x_1(t) \Rightarrow y_1(t) = e^{x_1(t)}$$
$$x_2(t) \Rightarrow y_2(t) = e^{x_2(t)}$$

Let us define $x_3(t)$ such that

$$x_3(t) = ax_1(t) + bx_2(t)$$

then the output becomes,

$$y_3(t) = e^{x_3(t)}$$
$$y_3(t) = e^{[ax_1(t)+bx_2(t)]}$$
$$y_3(t) = e^{ax_1(t)}e^{bx_2(t)}$$
$$y_3(t) \neq ay_1(t) + by_2(t)$$

Therefore, the system is nonlinear.

Problem 2.52 Determine whether the following discrete-time systems are linear or not.

(i) $y(n) = \ln[x(n)]$ (ii) $y(n) = x(n) - x(n-1)$

(iii) $y(n) = x^2(n) + x^2(n-1)$ (iv) $y(n) = 2x(n) + 4$

Solution

(i) $y(n) = \ln[x(n)]$

Let us define $x_1(n)$ and $x_2(n)$ as

$$x_1(n) \Rightarrow y_1(n) = \ln[x_1(n)]$$
$$x_2(n) \Rightarrow y_2(n) = \ln[x_2(n)]$$

Let us define $x_3(n)$ such that

$$x_3(n) = ax_1(n) + bx_2(n)$$

then the output becomes,

$$y_3(n) = \ln[x_3(n)]$$
$$y_3(n) = \ln[ax_1(n) + bx_2(n)]$$
$$y_3(n) \neq ay_1(n) + by_2(n)$$

Therefore, the system is nonlinear.

(ii) $y(n) = x(n) - x(n-1)$

Let us define $x_1(n)$ and $x_2(n)$ as

$$x_1(n) \Rightarrow y_1(n) = x_1(n) - x_1(n-1)$$
$$x_2(n) \Rightarrow y_2(n) = x_2(n) - x_2(n-1)$$

Let us define $x_3(n)$ such that

$$x_3(n) = ax_1(n) + bx_2(n)$$

then the output response becomes,

$$y_3(n) = x_3(n) - x_3(n-1)$$
$$y_3(n) = [ax_1(n) + bx_2(n)] - [ax_1(n-1) + bx_2(n-1)]$$
$$y_3(n) = a[x_1(n) - x_1(n-1)] + b[x_2(n) - x_2(n-1)]$$
$$y_3(n) = ay_1(n) + by_2(n)$$

Therefore, the system is linear.

(iii) $y(n) = x^2(n) + x^2(n-1)$

Let us define $x_1(n)$ and $x_2(n)$ as

$$x_1(n) \Rightarrow y_1(n) = x_1^2(n) - x_2^2(n-1)$$
$$x_2(n) \Rightarrow y_2(n) = x_2^2(n) - x_2^2(n-1)$$

Let us define $x_3(n)$ such that

$$x_3(n) = ax_1(n) + bx_2(n)$$

then the output becomes,

$$y_3(n) = x_3^2(n) + x_3^2(n-1)$$
$$y_3(n) = [ax_1(n) + bx_2(n)]^2 + [ax_1(n-1) + bx_2(n-1)]^2$$
$$y_3(n) = [a^2x_1^2(n) + b^2x_2^2(n) + 2ab\,x_1(n)x_2(n)]$$
$$\qquad + [a^2x_1^2(n-1) + b^2x_2^2(n-1) + 2ab\,x_1(n-1)x_2(n-1)]$$
$$y_3(n) = a^2\left[x_1^2(n) + x_1^2(n-1)\right] + b^2\left[x_2^2(n) + x_2^2(n-1)\right]$$
$$\qquad + 2ab[x_1(n)x_2(n) + x_1(n-1)x_2(n-1)]$$
$$y_3(n) \neq ay_1(n) + by_2(n)$$

Therefore, the system is nonlinear.

(iv) $y(n) = 2x(n) + 4$

Let us define $x_1(n)$ and $x_2(n)$ as

$$x_1(n) \Rightarrow y_1(n) = 2x_1(n) + 4$$
$$x_2(n) \Rightarrow y_2(n) = 2x_2(n) + 4$$

Let us define $x_3(n)$ such that

$$x_3(n) = ax_1(n) + bx_2(n)$$

then the output response becomes,

$$y_3(n) = 2x_3(n) + 4$$
$$y_3(n) = 2[ax_1(n) + bx_2(n)] + 4$$
$$y_3(n) = 2ax_1(n) + 2bx_2(n) + 4$$
$$y_3(n) \neq ay_1(n) + by_2(n)$$

Therefore, the system is nonlinear.

Problem 2.53 Determine whether the following continuous-time systems are linear or not.

(i) $\dfrac{d^2y(t)}{dt^2}+2\dfrac{dy(t)}{dt}+y(t)=\dfrac{dx(t)}{dt}+x(t)$

(ii) $\dfrac{d^2y(t)}{dt^2}+t\dfrac{dy(t)}{dt}+y(t)=t\dfrac{dx(t)}{dt}+x(t)$

Solution

(i) $\dfrac{d^2y(t)}{dt^2}+2\dfrac{dy(t)}{dt}+y(t)=\dfrac{dx(t)}{dt}+x(t)$

Let us define $x_1(t)$ and $x_2(t)$ as

$$x_1(t)\Rightarrow\frac{d^2y_1(t)}{dt^2}+2\frac{dy_1(t)}{dt}+y_1(t)=\frac{dx_1(t)}{dt}+x_1(t)$$

$$x_2(t)\Rightarrow\frac{d^2y(t)}{dt^2}+2\frac{dy_2(t)}{dt}+y_2(t)=\frac{dx_2(t)}{dt}+x_2(t)$$

Let us define $x_3(t)$ such that

$$x_3(t)=ax_1(t)+bx_2(t)$$

then the output response becomes,

$$\frac{d^2y_3(t)}{dt^2}+2\frac{dy_3(t)}{dt}+y_3(t)=\frac{dx_3(t)}{dt}+x_3(t)$$

$$\frac{d^2y_3(t)}{dt^2}+2\frac{dy_3(t)}{dt}+y_3(t)=\frac{d}{dt}[ax_1(t)+bx_2(t)]+[ax_1(t)+bx_2(t)]$$

$$\frac{d^2y_3(t)}{dt^2}+2\frac{dy_3(t)}{dt}+y_3(t)=a\frac{dx_1(t)}{dt}+b\frac{dx_2(t)}{dt}+ax_1(t)+bx_2(t)$$

$$\frac{d^2y_3(t)}{dt^2}+2\frac{dy_3(t)}{dt}+y_3(t)=a\left[\frac{dx_1(t)}{dt}+x_1(t)\right]+b\left[\frac{dx_2(t)}{dt}+x_2(t)\right]$$

Therefore, the system is linear.

(ii) $\dfrac{d^2y(t)}{dt^2}+t\dfrac{dy(t)}{dt}+y(t)=t\dfrac{dx(t)}{dt}+x(t)$

Let us define $x_1(t)$ and $x_2(t)$ as

$$x_1(t)\Rightarrow\frac{d^2y_1(t)}{dt^2}+t\frac{dy_1(t)}{dt}+y_1(t)=t\frac{dx_1(t)}{dt}+x_1(t)$$

$$x_2(t)\Rightarrow\frac{d^2y_2(t)}{dt^2}+t\frac{dy_2(t)}{dt}+y_2(t)=t\frac{dx_2(t)}{dt}+x_2(t)$$

Let us define $x_3(t)$ such that

$$x_3(t) = a\,x_1(t) + b\,x_2(t)$$

then the output response becomes,

$$\frac{d^2 y_3(t)}{dt^2} + t\frac{dy_3(t)}{dt} + y_3(t) = t\frac{dx_3(t)}{dt} + x_3(t)$$

$$= t\frac{d}{dt}[ax_1(t) + bx_2(t)] + [ax_1(t) + bx_2(t)]$$

$$= at\left[\frac{dx_1(t)}{dt}\right] + bt\left[\frac{dx_2(t)}{dt}\right] + ax_1(t) + bx_2(t)$$

$$\frac{d^2 y_3(t)}{dt^2} + t\frac{dy_3(t)}{dt} + y_3(t) = a\left[t\frac{dx_1(t)}{dt} + x_1(t)\right] + b\left[t\frac{dx_2(t)}{dt} + x_2(t)\right]$$

Therefore, the system is linear.

2.6.7 Causal and Noncausal System

Causal system In a causal system, the output response of the system at any time depends only on the present input and/or on the past input, but not on the future inputs.

In a causal system, the next input cannot be predicted. Hence, this may not be an essential condition for all systems.

Examples for causal system are

$$y(n) = x(n) - x(n-1)$$
$$y(t) = tx(t)$$

In the noncausal system, the output response of the system depends on the future input values also.

Examples for noncausal system are

$$y(n) = n^2 x(n)$$
$$y(t) = x(n+1) - x(n)$$

SOLVED PROBLEMS

Problem 2.54 Check whether the following systems are causal or not.

(i) $y(t) = tx(t)$ (ii) $y(t) = x(t^2)$ (iii) $y(t) = x^2(t)$

(iv) $y(t) = x(\sin t)$ (v) $y(t) = \int_{-\infty}^{4t} x(\tau)\,d\tau$ (vi) $y(t) = \frac{dx(t)}{dt}$

Solution

(i) $y(t) = tx(t)$

Let $t = 0$, then $y(0) = 0$

$t = 1$, then $y(1) = x(1)$

$t = -1$, then $y(-1) = -x(-1)$

For all values of t, the output depends on present and past values of the input.

Hence, the system is causal.

(ii) $y(t) = x(t^2)$

Let $t = 0$, then $y(0) = x(0)$

$t = 1$, then $y(+1) = x(1)$

$t = -1$, then $y(-1) = x(1)$

$t = 2$, then $y(2) = x(4)$

In the last two cases, the output value depends on the future value of the input. Hence, the system is noncausal.

(iii) $y(t) = x^2(t)$

Let $t = 0$, then $y(0) = x^2(0)$

$t = -1$, then $y(-1) = x^2(-1)$

$t = 1$, then $y(1) = x^2(1)$

The output value does not depend on the future value of the input. Hence, the system is causal.

(iv) $y(t) = x(\sin t)$

Let $t = 0$, then $y(0) = x\,[\sin(0)]$

$t = -1$, then $y(-1) = x\,[\sin(-1)]$

$t = 1$, then $y(1) = x\,[\sin(1)]$

The output value does not depend on the future value of the input. Hence, the system is causal.

(v) $y(t) = \int\limits_{\tau=-\infty}^{4t} x(\tau)\,d\tau$

Let $t = 0$, then

$$y(0) = \int_{-\infty}^{0} x(\tau)\,d\tau = x(\tau)\Big|_{-\infty}^{0} = x(0) - x(-\infty)$$

$t = 1$, then

$$y(1) = \int_{-\infty}^{4} x(\tau)\,d\tau = x(\tau)\Big|_{-\infty}^{4} = x(4) - x(-\infty)$$

$t = -1$, then

$$y(-1) = \int_{-\infty}^{-4} x(\tau)\,d\tau = x(\tau)\Big|_{-\infty}^{-4} = x(-4) - x(-\infty)$$

In the second case, the output value depends on the future value of the input. Hence, the system is noncausal.

(vi) $y(t) = \dfrac{dx(t)}{dt}$

Let $t = 0$, then $y(0) = \dfrac{dx(0)}{dt}$

$t = 1$, then $y(1) = \dfrac{dx(1)}{dt}$

$t = -1$, then $y(-1) = \dfrac{dx(-1)}{dt}$

The output value does not depend on the future values of the input. Hence, the system is causal.

Problem 2.55 Check whether the following systems are causal or not.

(i) $y(n) = x(n) - x(n-1)$ (ii) $y(n) = x(n^{1/2})$ (iii) $y(n) = \sum_{n=0}^{N-1} x(n)$

(iv) $y(n) = \sum_{n=-\infty}^{N-1} x(n)$ (v) $y(n) = x(n)x(n-1)$

Solution

(i) $y(n) = x(n) - x(n-1)$

Let $n = 0$, then $y(0) = x(0) - x(-1)$

$n = 1$, then $y(1) = x(1) - x(0)$

$n = -1$, then $y(-1) = x(-1) - x(-2)$

The output value depends on the present and past values of the input but not on the future value. Hence, the system is causal.

(ii) $y(n) = x(n^{1/2})$

Let $n = 0$, then $y(0) = x(0)$

$n = 1$, then $y(1) = x(1)$

$n = -1$, then $y(-1) = x[(-1)^{1/2}]$

$n = 4$, then $y(4) = x(2)$

The output value depends on present or past values of the input, but not on the future value. Hence, the system is causal.

(iii) $y(n) = \sum_{k=0}^{n-1} x(k)$

$$y(n) = \sum_{k=0}^{n-1} x(k) = x(0) + x(1) + x(2) + ... + x(n-1)$$

The output depends on the present and past values. Hence, the system is causal.

(iv) $y(n)=\sum_{k=-\infty}^{n-1} x(k)$

$$y(n)=\sum_{k=-\infty}^{n-1} x(k)= \dots x(-2)+x(-1)+x(0)+x(1)+x(2)+\dots x(n-1)$$

The output depends on the future values of the input. Hence, the system is noncausal.

(v) $y(n)=x(n)x(n-1)$

Let $n=0$, then $y(0)=x(0)x(-1)$

$n=1$, then $y(1)=x(1)x(0)$

$n=-1$, then $y(-1)=x(-1)x(-2)$

The output depends on the past and present values of the input. Hence, the system is causal.

2.7 INTERCONNECTION OF SYSTEMS

Interconnection of systems is considered to be one of the most important aspects in system design. In general, the actual system is always divided into many subsystems which are interconnected such that the actual work will be executed smoothly. Computing a larger systems involves longer time duration and increased complexity to analyse the system. When the system is divided into smaller modules, the complexity involved in the smaller module is lesser than considering the entire system at time. Due to the advancement of parallel computation concept, the module based system analysis work faster than considering entire system.

Following are the general configurations, in which any subsystem can be connected.

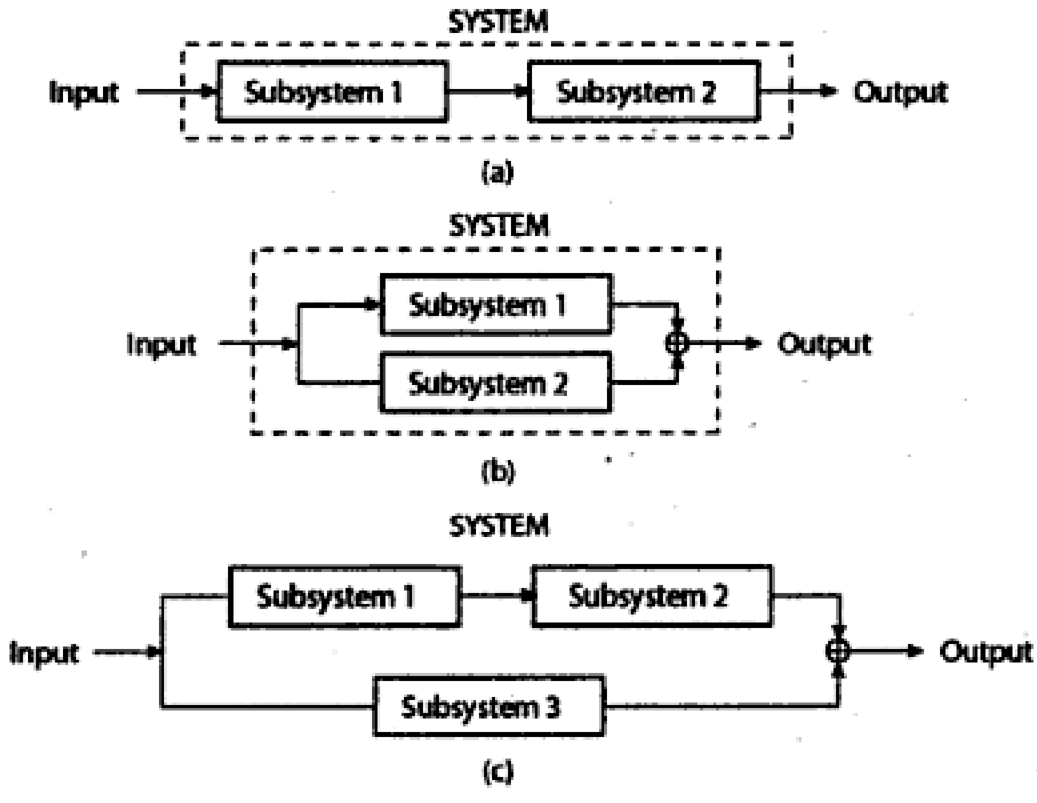


Fig. 2.55 Interconnection of Systems (a) Series (Cascade) Connection (b) Parallel Connection (c) Series–Parallel Connection

Fig. 2.55(a) illustrate the subsystems connected in series. Fig. 2.55(b) illustrate the subsystems connected in parallel. A serial-parallel combination of subsystems is shown in Fig. 2.55(c).

Let us consider a general communication receiver system, as shown in Fig. 2.56.

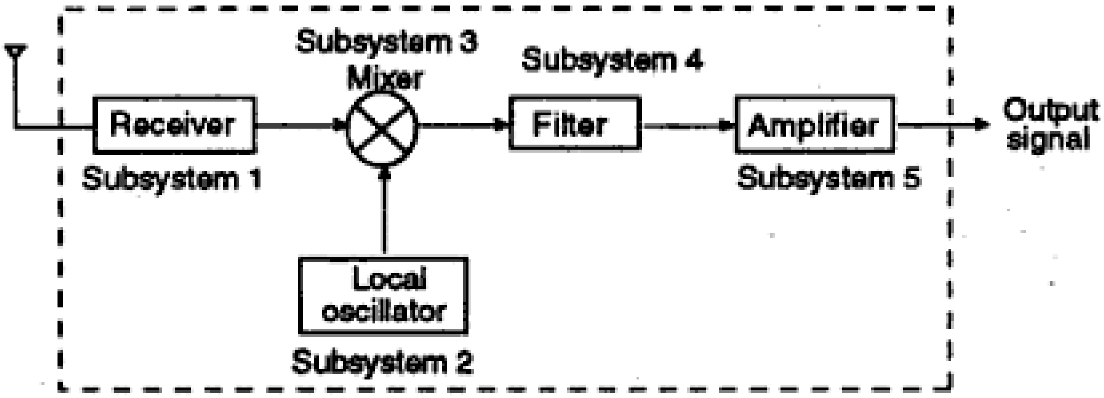


Fig. 2.56 Receiver of a Communication System

The receiver system consists of many subsystems like receiver (subsystem 1), local oscillator (subsystem 2) mixer (subsystem 3), filter (subsystem 4), and amplifier (subsystem 5). All these subsystems are connected either in parallel or in series, in order to execute the assigned process. If the systems are not connected in proper configuration (series or parallel), then the output may not be the expected one, though each and every subsystem is individually working.

CHAPTER SUMMARY

- A signal is defined as a function of one or more variables, which conveys information.
- A system is an entity that manipulates one or more input signals to perform a function, which results in a new output signal.
- The signals can be classified as
  - Continuous-time (CT) signal and Discrete-time (DT) signal
  - Periodic and Aperiodic signals
  - Even and Odd signals
  - Deterministic and Random signals
  - Energy and Power signals

- Differentiate discrete-time signal and digital signal

Discrete-time Signal	Digital Signal
A DT signal is obtained by sampling a CT signal at a uniform or non-uniform rate.	A digital signal is obtained by sampling, quantizing and encoding a CT signal.
A signal $x(n)$ is said to be DT signal if it defines or represents an input at discrete instants of time.	A signal is said to be a digital signal if it is represented in terms of binary bits ('0' or '1').
The DT signal is discrete in time only.	The digital signal is discrete in time and quantized in amplitude.
For a DT signal, the amplitude varies at every discrete values of 'n'.	For a digital signal, the amplitude is represented as a high voltage if the bit is '1' and low voltage if the bit is '0'.

- Differentiate energy and power signal.

Energy Signal	Power Signal
The energy of CT signal $x(t)$ over a period $-\frac{T}{2} \leq t \leq +\frac{T}{2}$ is given by $E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T/2}^{+T/2} \lvert x(t)\rvert^2\, dt$	The power of CT signal $x(t)$ over a period $-\frac{T}{2} \leq t \leq +\frac{T}{2}$ is given by $P = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{+T/2} \lvert x(t)\rvert^2\, dt$
The energy of DT signal $x(n)$ over a period $-N \leq n \leq +N$ is given by $E = \underset{N\to\infty}{\mathrm{Lt}} \sum_{n=-N}^{+N} \lvert x(n)\rvert^2$	The power of DT signal $x(n)$ over a period $-N \leq n \leq +N$ is given by $E = \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N+1}\sum_{n=-N}^{+N} \lvert x(n)\rvert^2$
A signal is referred to as energy signal if and only if the total energy of the signal satisfies the condition $0 \leq E_\infty \leq \infty$.	A signal is referred to as energy signal if and only if the total energy of the signal satisfies the condition $0 \leq P_\infty \leq \infty$.
Generally deterministic and aperiodic signals are considered to be energy signals.	Generally random and periodic signals are considered to be power signals.

- Differentiate even and odd signals.

Odd Signal	Even Signal
A signal is said to be odd if $x(t) = -x(-t)$ for CT signal $x(n) = -x(-n)$ for DT signal	A signal is said to be even if $x(t) = x(-t)$ for CT signal $x(n) = x(-n)$ for DT signal
The odd component of any signal is $x_0(t) = \frac{x(t) - x(-t)}{2}$ for CT signal $x_0(n) = \frac{x(n) - x(-n)}{2}$ for DT signal	The even component of any signal is $x_e(t) = \frac{x(t) + x(-t)}{2}$ for CT signal $x_e(n) = \frac{x(n) + x(-n)}{2}$ for DT signal
Odd signals are anti-symmetric about the vertical axis Example: Sine wave	Even signals are symmetric about the vertical axis Example: Cosine wave

- Differentiate random and deterministic signal.

Deterministic Signal	Random Signal
A deterministic signal is one in which there is a certainty with respect to its values at any time.	A random signal is one in which there is a uncertainty with respect to its values at any time.
Future value of signals is predictable. Eg. Pulse train, sinusoidal wave etc.	Future value of signals is unpredictable. Eg. EEG signal, Noise, Speech etc.
Deterministic signals can be expressed mathematically.	Random signals are expressed mathematically in terms of impulses.

- The fundamental period of N_0 of a DT signal $x(n)$ is the smallest positive value of N for which $x(n) = x(n+N)$ or the DT signal exhibits periodicity. It is defined as $N_0 = \frac{2\pi}{\Omega_0} m$ where m is an integer value.
- The fundamental period T_0 of a CT signal $x(t)$ is the smallest positive value of T for which $x(t) = x(t+T)$ the CT signal exhibits periodicity. It is defined as $T_0 = \frac{2\pi}{\omega_0}$
- Basic operations performed on dependent variables of signal
  - Amplitude scaling
  - Addition of signals
  - Multiplication of signals
  - Differentiation of signals
  - Integration of signals

- Basic operations performed on independent variables of signal
  - Time scaling
  - Reflection of signal
  - Time shifting
- Amplitude scaling is considered as the multiplication of scalar value α with continuous-time signal $x(t)$ [$x(n)$ for discrete-time signal], that is, $\alpha\, x(t)$, [$\alpha\, x(n)$ for discrete-time signal]. If the value of $\alpha > 1$, then signal is said to be amplified. If the value of $\alpha < 1$, then signal is said to be attenuated. During amplitude scaling the time scale remains the same.
- When any two signals are added, their amplitude will be changed but the time scale remains unaltered.
- When any two signals are multiplied, their amplitude will be changed but the time scale remains unaltered.
- When any signal is differentiated, their amplitude will be changed but the time scale remains unaltered.
- When any signal is integrated, their amplitude will be changed but the time scale remains unaltered.
- Time scaling is considered as the multiplication of scalar value α with the time function of continuous-time signal $x(t)$ [$x(n)$ for discrete-time signal], that is, $x(\alpha t)$ [$x(\alpha n)$ for discrete-time signal]. If the value of $\alpha > 1$, then signal is said to be compressed. If the value of $\alpha < 1$, then signal is said to be expanded. During time scaling the amplitude remains the same.
- A continuous-time signal $x(t)$ [$x(n)$ for discrete-time signal] is said to be shifted right side, if it satisfies the condition $x(t-t_0)$ [$x(n-n_0)$ for discrete-time signal] and shifted left side, if it satisfies the condition $x(t+t_0)$ [$x(n+n_0)$ for discrete-time signal]. A continuous-time signal $x(-t)$ [$x(-n)$ for discrete-time signal] is said to be shifted right side, if it satisfies the condition $x(t+t_0)$ [$x(n+n_0)$ for discrete-time signal] and shifted left side, if it satisfies the condition $x(t-t_0)$ [$x(n-n_0)$ for discrete-time signal].
- A real continuous-time exponential signal in its more general form represented as $x(t) = Be^{\alpha t}$ where B is the real scaling factor and α is the real parameter. For $\alpha < 0$ the magnitude of the continuous-time exponential signal decays exponentially. For $\alpha > 1$ the magnitude of the continuous-time exponential signal rises exponentially.
- A real discrete-time exponential signal in its more general form represented as $x(n) = B\alpha^n$ where B is the real scaling factor and α is the real parameter. For $\alpha > 1$ the magnitude of the discrete-time exponential signal rises exponentially. For $1 > \alpha > 0$ the magnitude of the discrete-time exponential signal decays exponentially.
- The continuous-time step function is denoted by $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
- The continuous-time step function is denoted by $u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$
- The impulse function is denoted by $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$
- The continuous-time ramp function is denoted by $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$
- The discrete-time ramp function is denoted by $r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

- The system can be classified as
  - Continuous-time and Discrete-time systems
  - Stable and Unstable systems
  - Memory and Memoryless systems
  - Invertible and Non-invertible systems
  - Time-invariant and Time-variant systems
  - Linear and Non-linear systems
  - Causal and Non-causal systems
- If the input and output of the system are continuous-time signals, then the system is called "continuous-time system".
- If the input and output of the system are discrete-time signals, then the system is called "discrete-time system".
- A system is said to be stable if and only if every bounded input produces a bounded output.
- A system is said to possess memory if the output of the system depends on past and future values.
- A system is said to be an invertible system if the input signal given to the system can be recovered from the output signal of the system.
- A system is said to be time-invariant if the input signal is delayed or advanced by any factor that leads to same delay or advancement respectively in the output time scale by the same factor, that is, the system responds to an input at any instant of time and results in an output.
- A system is said to be linear if it satisfies superposition, scaling and additive property, that is, the response to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$ (additive property) and the response to $ax_1(t) + bx_2(t)$ is $ay_1(t) + by_2(t)$ (scaling property), where a and b are complex constants. The same definition holds good for discrete-time system also.
- A system is said to be causal if the output response of the system at any time depends only on the present input and/or on the past input, but not on the future inputs.

REVIEW QUESTIONS

1. Define signal.
2. Explain one-dimensional signal with suitable examples.
3. Explain two-dimensional signal with suitable examples.
4. Explain multi-dimensional signal.
5. Distinguish between continuous-time signal and discrete-time signal.
6. What is the basic difference between discrete-time signal and digital signal?
7. How do you classify signals?
8. Differentiate energy signal and power signal.
9. Differentiate even signal and odd signal.
10. Explain periodic signal.

11. Differentiate deterministic signal and random signal.
12. Test whether the following signals are periodic or not. If yes, what is its fundamental period?

(a) $x(n) = \sin^2\left(\frac{\pi}{4}\right)n\, u(n)$ (b) $x(n) = be^{\left(\frac{j4\pi(n+1)/4}{5}\right)}$

(c) $x(t) = e^{(-2+4j)t}$ (d) $x(t) = \left[\sin\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)\right]^2$

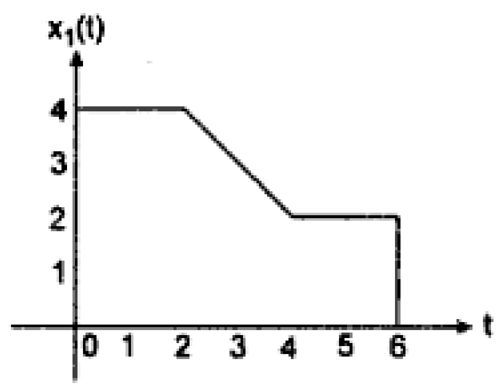
(e) $x(n) = \cos\left(\frac{\pi}{3}\right)n + \sin\left(\frac{\pi}{3}\right)n$

13. Find the odd and even components of the following signals.

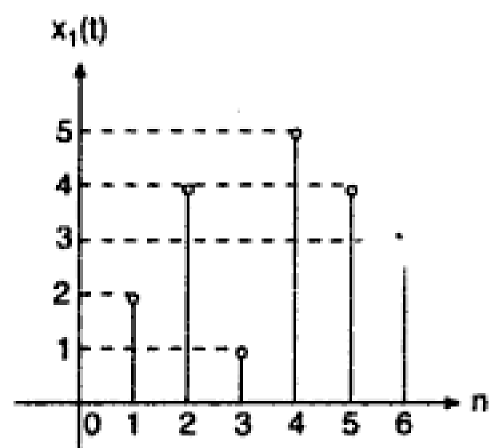
(a) $x(n) = \{2, 4, 6, 8, 10\}$ (b) $x(t) = e^{-j(\omega t+\phi)}$

(c) $x(t) = \sin 2t + \cos 2t$ (d) $x(n) = e^{j2\omega n}$

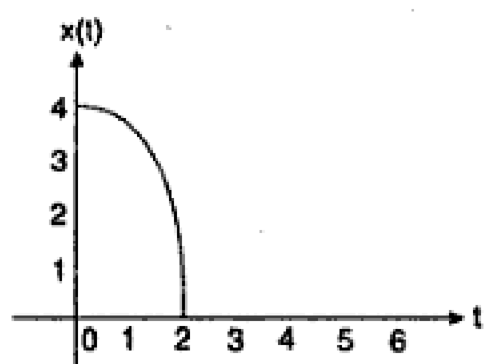
14. Draw the odd and even components of the given signal.



(a)



(b)



(c)

15. Test whether the following signals are energy signals or power signals.

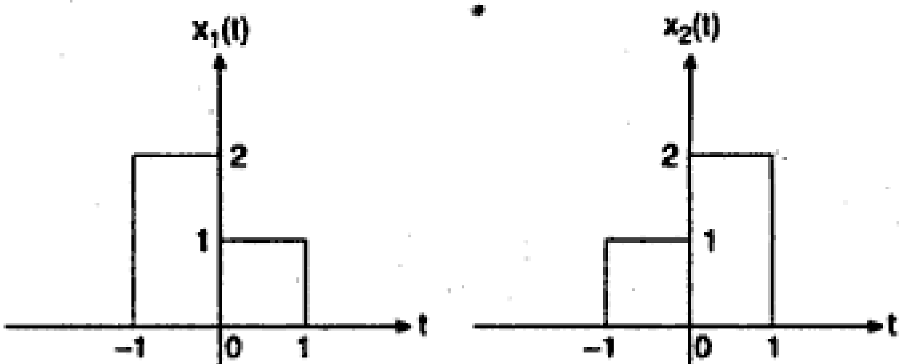
(a) $x(t) = \sin 2t\, u(t)$ (b) $x(t) = t^2\, u(t)$

(c) $x(t) = \sin 3t$ (d) $x(n) = e^{j\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)}u(n)$

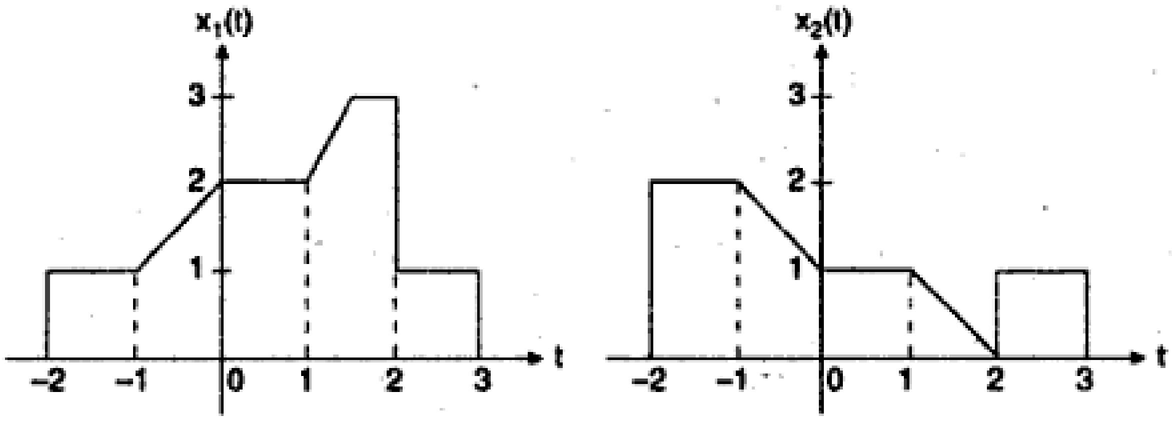
(e) $x(n) = u(n) - u(n-6)$ (f) $x(t) = e^{-j\left(\frac{\pi}{6}t + \frac{\pi}{3}\right)}u(t)$

(g) $x(n) = \cos 4n\, u(n)$ (h) $x(n) = \left(\frac{1}{3}\right)^n u(n)$

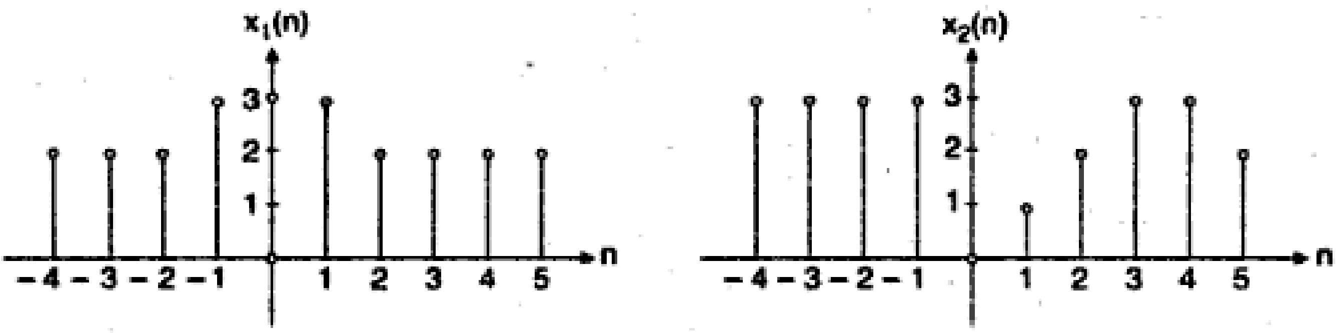
16. **Perform addition and multiplication on the following signals.**



(a)

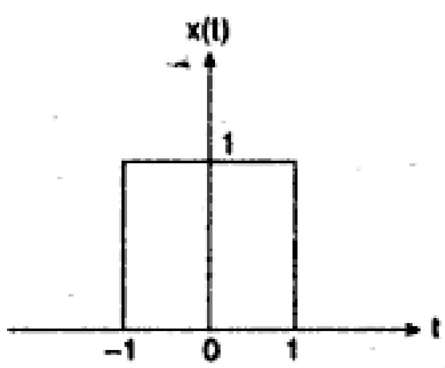


(b)



(c)

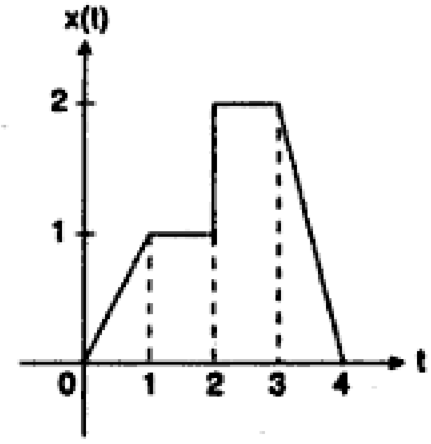
17. **Find $y(t)$ for the given signal.**



(a) $y(t) = x(5t+6)$ (b) $y(t) = x(5t-6)$

(c) $y(t) = x\left(\dfrac{5t-6}{10}\right)$ (d) $y(t) = x\left(\dfrac{5t+6}{10}\right)$

18. Find $y(t)$ for the given signal.



(a) $y(t) = x(3t+4)$ (b) $y(t) = x(3t-4)$

(c) $y(t) = x\left(\dfrac{3t-4}{9}\right)$ (d) $y(t) = x\left(\dfrac{3t+4}{9}\right)$

19. Define the following signal mathematically and represent graphically.

(a) Impulse signal (b) Ramp signal

(c) Step signal (d) Sinusoidal signal

(e) Exponential signal with various time periods

20. Sketch the continuous-time signal $x(t) = 10\cos(2\pi t)$ for the interval $2 \geq t \geq 0$ and sketch the corresponding discrete-time signal with a sampling period $T = 0.1$ s.

21. Sketch the continuous-time signal $x(t) = e^{-t}$ for the interval $2 \geq t \geq -2$ and sketch the corresponding discrete-time signal with a sampling period $T = 0.2$ s.

22. Define a system with suitable examples.

23. Give a broad classification of systems.

24. Define stability of a system.

25. Test whether the following systems are stable or not.

(a) $h(n) = b^n u(-n)$ (b) $h(n) = 4^{n-1} u(n-2)$

(c) $h(t) = t^2 e^{at} u(t)$ (d) $h(t) = t \sin t \, u(t)$

(e) $h(t) = e^{-j2t} u(t)$ (f) $h(n) = e^{n/3} u(n-6)$

26. Define a memory system with suitable examples.

27. Test whether the following are memory systems or not.

(a) $y(t) = x(t) + x(t+1)$ (b) $y(t) = \dfrac{d^2 x(t)}{dt^2}$

(c) $y(t) = x(n-1) + x(1-n)$ (d) $y(t) = tx(t)$

(e) $y(t) = x^3(t)$ (f) $y(n) = e^{n} u(n)$

28. Define a time-invariant system with suitable examples.
29. Test whether the following systems are time-invariant or not.

(a) $y(t) = \cos[x(t)]$ (b) $y(n) = n\,x(n)$

(c) $y(n) = x(n)\cos\omega n$ (d) $y(n) = e^{x(n)}$

(e) $y(t) = y(t-1) + 2t\,y(t-2)$ (f) $\frac{dy(t)}{dt} + 2y(t) = x(t)$

(g) $\frac{d^2y(t)}{dt^2} + 3t\,y(t) = x(t)$ (h) $y(n) = \log_{10}|x(n)|$

30. Define a linear system with suitable examples.
31. Test whether the following systems are stable or not.

(a) $y(n) = n^2x(n)$

(b) $y(n) = n\,x(n-2) + k$

(c) $y(t) = e^{2x(t)} + k$

(d) $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$

(e) $\frac{d^2y(t)}{dt^2} + 4t\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$

32. Define causality with suitable examples.
33. Test whether the following systems are causal or not.

(a) $y(n) = nx(n)$ (b) $y(n) = x(\cos 2n)$

(c) $y(n) = x^2(n)$ (d) $y(t) = \frac{d^2x(t)}{dt^2} + x(t)$

(e) $y(t) = x(t)x(t-2)$ (f) $y(t) = \int_{-2}^{t} x(\tau)$

34. Explain the various configurations in which systems are connected, justify their advantages and mention their applications.
35. Show whether the system $y(n) \le n\,x(n)$ is a linear time-invariant system.

CHAPTER

3

LTI Systems

Linearity and time-invariant properties are considered to be important in analyzing and realizing a system. In this chapter, the relation between input and output that satisfies the linearity and time-invariant properties of the system are discussed. In this chapter, term "convolution sum" is introduced, which gives the mathematical relationship for the input-output. The input-output relation is explicitly discussed both in discrete-time and continuous-time.

■ 3.1 DISCRETE-TIME LINEAR TIME-INVARIANT SYSTEM

The impulse response is the output of the Linear Time-invariant (LTI) discrete system when the impulse $\delta(n)$ is applied to it.

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{elsewhere} \end{cases} \tag{3.1}$$

The impulse response completely characterizes the behaviour of any LTI system. The discrete-time unit impulse can be used to construct any discrete-time signal.

3.1.1 Representation of Discrete-time Signals in Terms of Impulses

Let us consider the product of a signal $x(n)$ and the impulse sequence $\delta(n)$, written as

$$x(n)\delta(n) = x(n)\delta(0) \tag{3.2}$$

From equation (3.1), it is clear that the impulse sequence exists only at $n = 0$. Therefore, $x(n)\delta(n)$ can be rewritten as $x(n)\delta(0)$. Though the impulse sequence exists at $n = 0$, the input signal $x(n)$ exists in the remaining samples.

Similarly,

$$\begin{aligned}&x(-2)\delta(n+2)\\&x(-1)\delta(n+1)\\&x(0)\delta(n)\\&x(1)\delta(n-1)\\&x(2)\delta(n-2)\end{aligned}$$

Therefore, the generalized relationship between the input signal $x(n)$ and the shifted impulse sequence is given by

$$x(n)\delta(n-k) = x(k)\delta(n-k) \tag{3.3}$$

In equation (3.3), $x(n)$ represents the input signal but $x(k)$ represents the magnitude of the input signal $x(n)$ at time k. In equation (3.3), the product of input signal $x(n)$ and time-shifted impulse $\delta(n-k)$ results in the time-shifted impulse $\delta(n-k)$ whose magnitude is the value of the signal $x(n)$ at time k, that is, $x(k)$ represents the magnitude of the signal at k and $\delta(n-k)$ represents the position of impulse signal.

Let us analyze the above statement graphically by considering a random signal as shown in Fig. 3.1.

It is clear from Fig. 3.1, that the signal $x(n)$ can be decomposed into the product of time-shifted impulse and signal $x(n)$ at k, i.e.

$$x(n) = \ldots + x(-4)\delta(n+4) + x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n)$$
$$+x(1)\delta(n-1) + x(2)\delta(n-2) + \ldots$$

$$x(n) = \sum_{k=-\infty}^{+\infty} x(k)\delta(n-k) \tag{3.4}$$

Any signal can be represented as a time-shifted impulse sequence $\delta(n-k)$.

If the input signal $x(n) = u(n)$, i.e.

$$u(n) = \begin{cases}1, & n \geq 0\\ 0, & \text{elsewhere}\end{cases}$$

then any unit step signal can be represented as

$$u(n) = \sum_{k=0}^{+\infty} \delta(n-k) \tag{3.5}$$

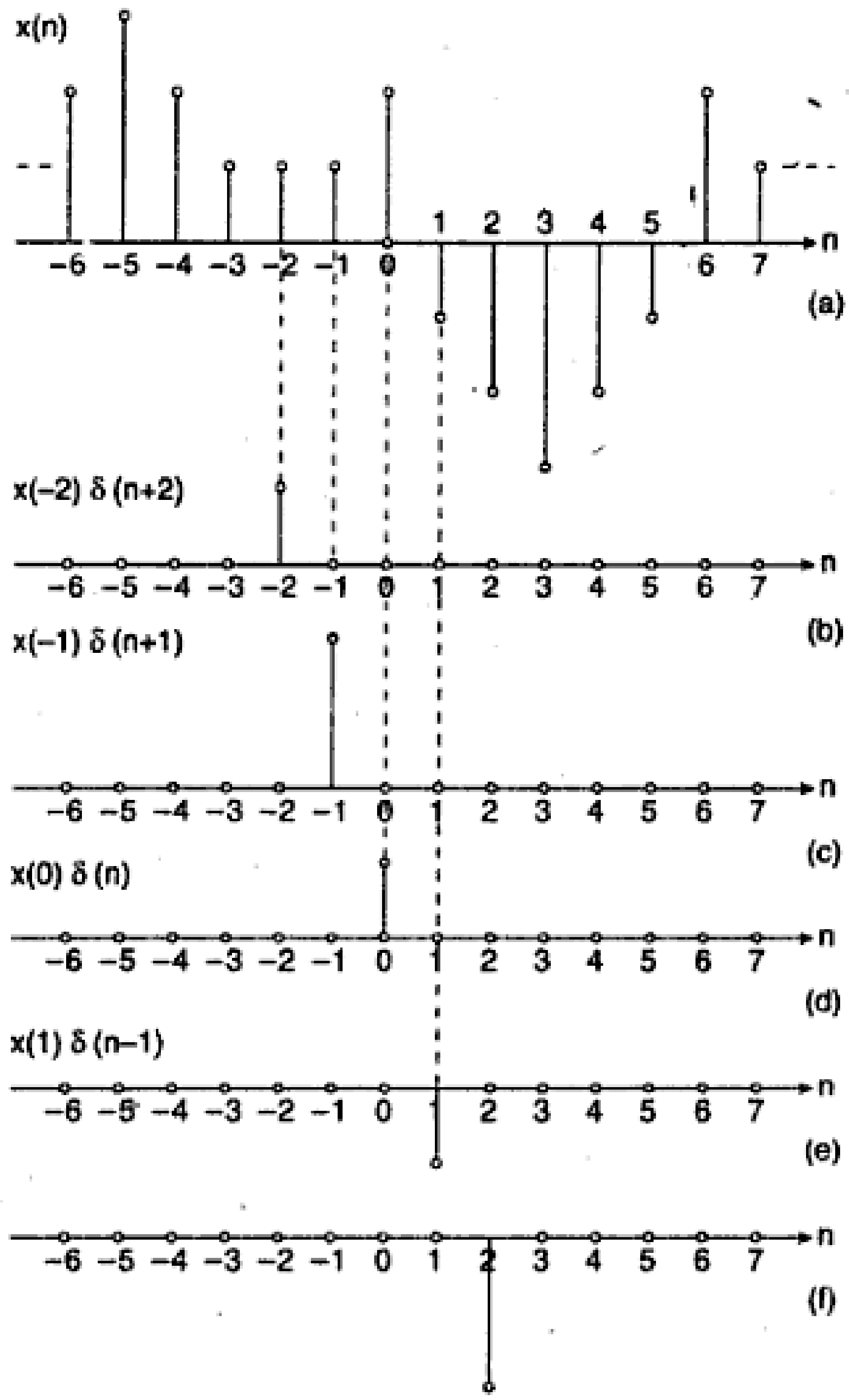


Fig. 3.1 Signal Decomposition

3.1.2 Convolution Sum

Let us consider a system, say H[•], to which the input signal $x(n)$ given in equation (3.4) is applied. The output signal becomes

$$y(n) = \mathrm{H}[x(n)] \tag{3.6}$$

$$y(n) = \mathrm{H}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

Fig. 3.2 System Representation

By linearity definition the term convolution describes, how the input signal interact with the system to produce the output signal. It is particularly useful to consider the output from the system owing to an impulse input. This is because any input signal may be represented as a sequence of impulses of different strengths. The operator $H[\cdot]$ operates only on function but not on magnitude of the signal $x(k)$. Therefore,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,H[\delta(n-k)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h_k(n) \tag{3.7}$$

where $h_k(n) = H[\delta(n-k)]$

If the system is an LTI (Linear Time-invariant) system, then the time-shifted input results in a time-shifted output, i.e.

$$h_k(n) = h_o(n-k) \tag{3.8}$$

The output due to the time-shifted impulse is a time-shifted version of the output due to an impulse.

Let $h(n) = h_0(n)$, then

$$h_k(n) = h(n-k) \tag{3.9}$$

Therefore, equation (3.7) can be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \tag{3.10}$$

The equation (3.10) is called the 'convolution sum' and is denoted by the symbol $*$, i.e.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = x(n) * h(n) \tag{3.11}$$

SOLVED PROBLEMS

Problem 3.1 Express the given signal sequence as a time-shifted impulse.

$$x(n) = \{1, -2, 8, \underset{\uparrow}{4}, 5, -3, 7\}$$

Solution In general, the arrow '↑' shows the value of the data for $n = 0$.

n	−3	−2	−1	0	1	2	3
x(n)	1	−2	8	4	5	−3	7

From equation (3.4), it is clear that any signal can be represented as linear combination of magnitude and position/type.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$x(n) = \sum_{k=-3}^{+3} x(k)\,\delta(n-k)$$

$$x(n) = x(-3)\,\delta(n+3) + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3)$$

$$x(n) = \delta(n+3) - 2\delta(n+2) + 8\delta(n+1) + 4\delta(n) + 5\delta(n-1) - 3\delta(n-2) + 7\delta(n-3)$$

Problem 3.2 Express the given signal sequence as time-shifted impulse.

$$x(n) = \{2,3,0,7,8,-15,18,20\}$$
$$\uparrow$$

Solution

n	−2	−1	0	1	2	3	4	5
$x(n)$	2	3	0	7	8	−15	18	20

From equation (3.4), it is clear that any signal can be represented as linear combination of magnitude and position/type.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$x(n) = \sum_{k=-2}^{5} x(k)\delta(n-k)$$

$$x(n) = x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1)$$
$$+x(2)\delta(n-2) + x(3)\delta(n-3) + x(4)\delta(n-4) + x(5)\delta(n-5)$$

$$x(n) = 2\delta(n+2) + 3\delta(n+1) + 7\delta(-1) + 8\delta(n-2) - 15\delta(n-3) + 18\delta(n-4) + 20\delta(n-5)$$

3.2 PROPERTIES OF LTI SYSTEM

3.2.1 Distributive Property

Let us consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected parallelly as shown in Fig. 3.3.

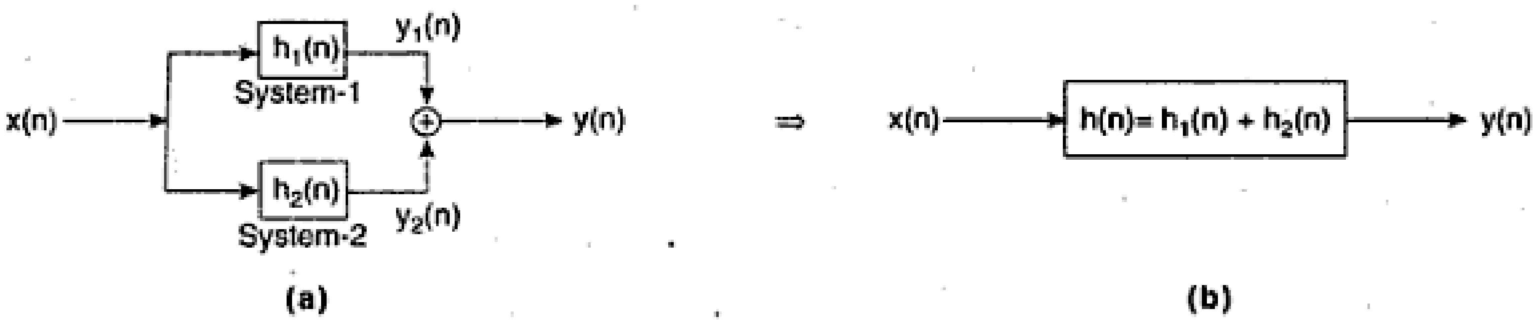


Fig. 3.3 (a) Systems Connected in Parallel (b) Equivalent Representation

By definition of the distributive property,

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n) \tag{3.12}$$

Proof The output of the first system is

$$y_1(n) = x(n) * h_1(n)$$

Similarly, the output of the second system is

$$y_2(n) = x(n) * h_2(n)$$

The overall output of the system $y(n)$ is given by

$$y(n) = y_1(n) + y_2(n)$$

$$y(n) = x(n) * h_1(n) + x(n) * h_2(n)$$

By the definition of convolution from equation (3.11)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k) h_2(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k) + h_2(n-k)] = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

where $\quad h(n-k) = h_1(n-k) + h_2(n-k) \Rightarrow h(n) = h_1(n) + h_2(n)$

Therefore, $\quad y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = x(n) * h(n)$

If two systems $h_1(n)$ and $h_2(n)$ are connected in parallel, then the impulse response of the system to the input signal $x(n)$ is equal to sum of the two impulse responses.

3.2.2 Associative Property

Let us consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in series as shown in Fig. 3.4.

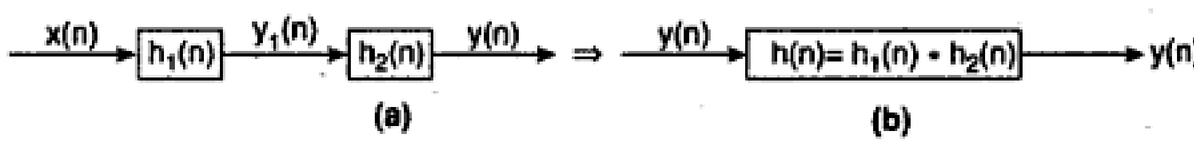


Fig. 3.4 (a) Systems Connected in Series (b) Equivalent Circuit

By definition of the associative property,

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)] \tag{3.13}$$

Proof The output of the first system

$$y_1(n) = x(n) * h_1(n) \tag{3.14}$$

Similarly, the output of the second system

$$y(n) = y_1(n) * h_2(n) \tag{3.15}$$

Substitute equation (3.14) in equation (3.15),

$$y(n) = [x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

3.2.3 Commutative Property

Let us consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in series as shown in Fig. 3.5.

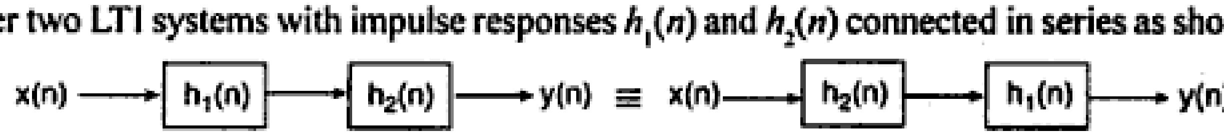


Fig. 3.5 Systems Connected in Series

By definition of the commutative property,

$$h_1(n)*h_2(n)=h_2(n)*h_1(n) \tag{3.16}$$

$$h_1(n)*h_2(n)=\sum_{k=-\infty}^{\infty} h_1(k)\,h_2(n-k)$$

Proof

Let $n-k=m \Rightarrow k=n-m$

$$h_1(n)*h_2(n)=\sum_{m=-\infty}^{\infty} h_1(n-m)h_2(m)$$

$$h_1(n)*h_2(n)=\sum_{m=-\infty}^{\infty} h_2(m)h_1(n-m)$$

$$h_1(n)*h_2(n)=h_2(n)*h_1(n)$$

Equation (3.16) finds application in solving convolution problems. This property can also be extended to signals, i.e.

$$y(n)=x(n)*h(n)=h(n)*x(n) \tag{3.17}$$

■ 3.3 PROPERTIES OF DISCRETE-TIME LTI SYSTEM

3.3.1 LTI System With and Without Memory

A system is memoryless if the output at any time depends only on the present input (discussed in Chapter 2). This is true for the LTI system if and only if

$$h(n)=0, \quad n\neq 0$$

Let us consider the impulse response of the form

$$h(n)=k\,\delta(n)$$

where $k=h(0)$, is a constant

The output of such a system is given by

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)\,k\delta(n-k)$$

$$y(n)=k\,x(k) \tag{3.18}$$

Hint	$\delta(n)=1,$	$n=0$
	$\delta(n-k)=1,$	$n=k$

Equation (3.18) is a memoryless LTI system.

If $h(n)\neq 0, n\neq 0$, then the LTI system is called a memory system.

3.3.2 Invertibility of LTI System

A system is invertible only if an inverse system exists. Similarly, an LTI system is invertible only if an inverse LTI system exists.

Let us consider the following figure.

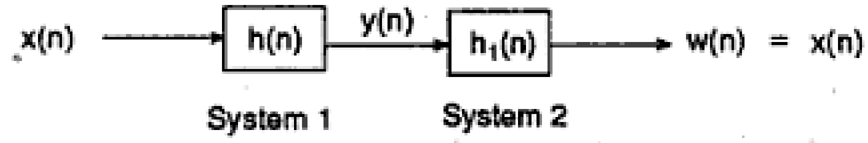


Fig. 3.6 Invertibility of LTI System

The system response $h(n)$ results in an output $y(n)$ and the output of system 1 is given to system 2, whose response $h_1(n)$ results in an output $w(n)$, which is equal to the original input $x(n)$. This is possible if

$$h(n) * h_1(n) = \delta(n) \tag{3.19}$$

3.3.3 Stability for LTI System

A system is said to be stable if every bounded input produces a bounded output. The statement can be extended to LTI systems also.

Let us consider a bounded input $x(n)$, i.e.

$$|x(n)| < M_X < \infty \text{ for all } n \tag{3.20}$$

Suppose the bounded input is applied to an LTI system with unit impulse response $h(n)$, then using convolution sum, we obtain an expression for the output $y(n)$, i.e.

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(n)\, h(n-k) \right| \tag{3.21}$$

By the inequality relation, the magnitude of the sum of a set of numbers is no longer larger than the sum of the magnitudes of the numbers, i.e.

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |x(k)||h(n-k)|$$

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \tag{3.22}$$

From equation (3.20), $\quad |x(n)| < M_X < \infty$

Therefore, $\quad |x(n-k)| < M_X < \infty$ for all n and k.

Substitute the equivalent relation of equation (3.20) in (3.22)

$$|y(n)| \le M_X \sum_{k=-\infty}^{\infty} |h(k)| \text{ for all } k \tag{3.23}$$

The impulse response $h(k)$ is absolutely summable if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{3.24}$$

then the output of the LTI system $y(n)$ is stable (bounded output). If the impulse response $h(k)$ is not absolutely summable, then the system is a 'nonstable system'.

SOLVED PROBLEMS

Problem 3.3 Find whether the system with impulse response $h(n) = 2e^{-2|n|}$ is stable or not.

Solution The condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty}|h(n)|=\sum_{n=-\infty}^{\infty}2e^{-2|-n|}=2\left(\sum_{n=-\infty}^{-1}e^{-2(-n)}+\sum_{n=0}^{\infty}e^{-2(n)}\right)$$

$$\sum_{n=-\infty}^{\infty}|h(n)|=2\left(\sum_{n=1}^{\infty}e^{-2n}+\sum_{n=0}^{\infty}e^{-2n}\right)$$

$$\sum_{n=-\infty}^{\infty}|h(n)|=2\left(\frac{e^{-2}}{1-e^{-2}}+\frac{1}{1-e^{-2}}\right)$$

$$\sum_{n=-\infty}^{\infty}|h(n)|=2\left(\frac{1+e^{-2}}{1-e^{-2}}\right)<\infty$$

Hint $\sum_{n=k}^{\infty}\beta^k=\frac{\beta^k}{1-\beta}, |\beta|<1$

$\sum_{n=0}^{\infty}\beta^k=\frac{1}{1-\beta}, |\beta|<1$

Therefore, the system is stable.

Problem 3.4 Find whether the system with impulse response $h(n) = e^{2n}\,u(n)$ is stable or not.

Solution The condition for stability is

$$\sum_{n=-\infty}^{\infty}|h(n)|<\infty$$

$$\sum_{n=-\infty}^{\infty}|h(n)|=\sum_{n=0}^{\infty}e^{2n}=1+e^2+e^4+e^6+\dots=\infty$$

Therefore, the system is unstable.

3.3.4 Causal System

By definition, for a discrete-time causal LTI system, the impulse response $h(n)$ must be zero for $n<0$. The causality can be extended to convolution sum as

$$y(n)=\sum_{k=-\infty}^{\infty}x(k)\,h(n-k) \tag{3.25}$$

For a causal discrete-time LTI system, $h(n)=0$ for $n<0$. Therefore, the output of a causal system must be expressed as

$$y(n)=\sum_{k=0}^{\infty}x(k)\,h(n-k) \tag{3.26}$$

A causal system cannot generate an output before an input is given to the system.

■ 3.4 LINEAR CONVOLUTION

The convolution equation defined in equation (3.10) can be given as an algorithm:

1. Plot both $x(k)$ and $h(k)$
2. Reflect $h(k)$ about $k=0$ to obtain $h(-k)$
3. Shift $h(-k)$ by n (towards left for $-n$ and towards right for $+n$)

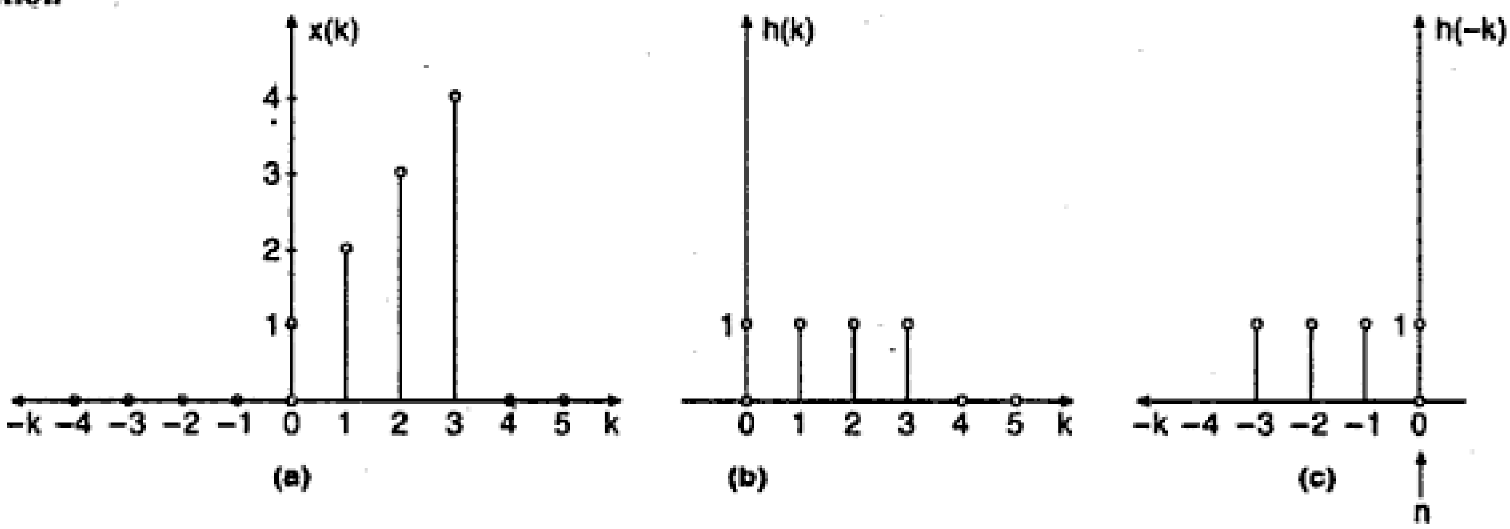
4. Let the initial value of n be negative
5. Multiply each element of $x(k)$ with $h(n-k)$ and add all the product terms to obtain $y(n)$
6. Shift $h(n-k)$ by incrementing the value of n by one, and do step 5
7. Do step 6 until the product of $x(k)$ and $h(n-k)$ reduces to zero

SOLVED PROBLEMS

Problem 3.5 Perform the convolution of the two sequences

$$x(n)=\{1,2,3,4\};\ h(n)=\{1,1,1,1\}$$

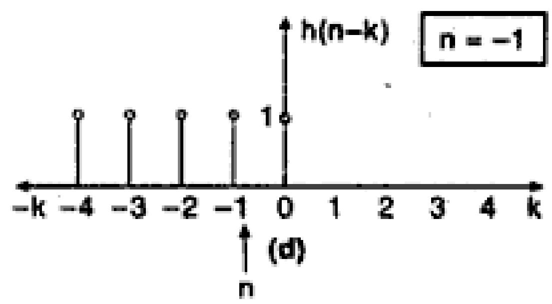
Solution



When $n=-1$

Multiply the elements of Fig. (a) and Fig. (d)

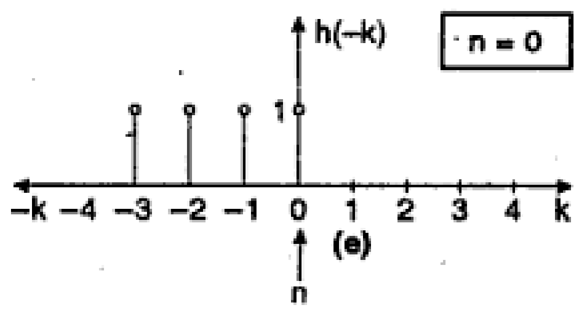
$$y(-1)=x(-4)\,h(-4)+x(-3)\,h(-3)+x(-2)\,h(-2)$$
$$+x(-1)\,h(-1)+x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)$$
$$y(-1)=0(1)+0(1)+0(1)+0(1)+1(0)+2(0)+3(0)+4(0)$$
$$y(-1)=0$$



When $n=0$

Multiply the elements of Fig. (a) and Fig. (e)

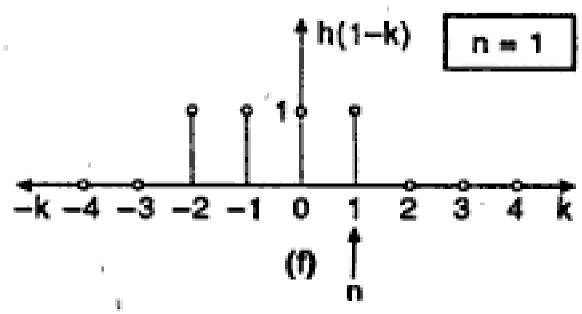
$$y(0)=x(-3)\,h(-3)+x(-2)\,h(-2)+x(-1)\,h(-1)+x(0)\,h(0)$$
$$+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)$$
$$y(0)=0(1)+0(1)+0(1)+0(1)+1(1)+2(0)+3(0)+4(0)$$
$$y(0)=1$$



When $n=1$

Multiply the elements of Fig. (a) and Fig. (f)

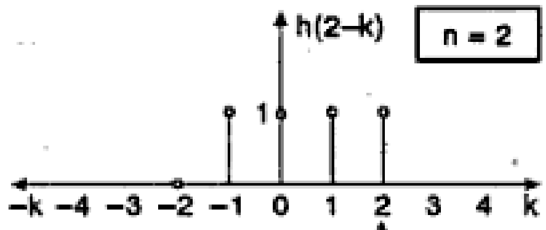
$$y(1)=x(-2)\,h(-2)+x(-1)\,h(-2)+x(0)\,h(0)$$
$$+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)$$
$$y(1)=0(1)+0(1)+1(1)+2(1)+3(0)+4(0)$$
$$y(1)=1+2=3$$



When $n=2$

Multiply the elements of Fig. (a) and Fig. (g)

$$y(2)=x(-1)\,h(-1)+x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)$$
$$y(2)=0(1)+1(1)+2(1)+3(1)+4(0)$$
$$y(2)=1+2+3=6$$

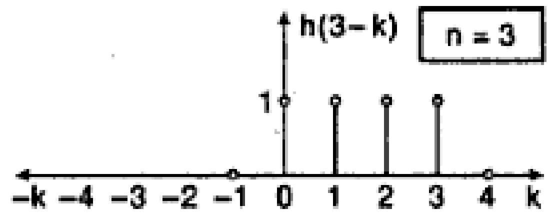


(g)

When $n=3$

Multiply the elements of Fig. (a) and Fig. (h)

$$y(3)=x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)$$
$$y(3)=1(1)+2(1)+3(1)+4(1)$$
$$y(3)=1+2+3+4=10$$

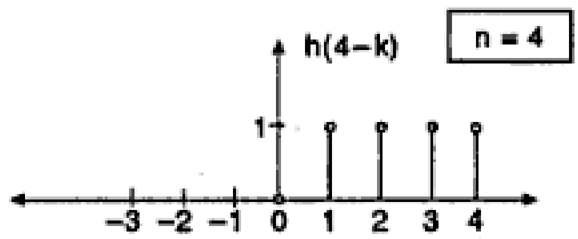


(h)

When $n=4$

Multiply the elements of Fig. (a) and Fig. (i)

$$y(4)=x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)+x(4)\,h(4)$$
$$y(4)=1(0)+2(1)+3(1)+4(1)+0(1)$$
$$y(4)=2+3+4=9$$

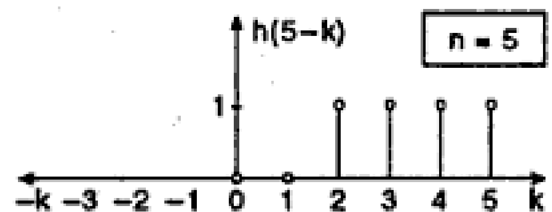


(i)

When $n=5$

Multiply the elements of Fig. (a) and Fig. (j)

$$y(5)=x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)+x(4)\,h(4)+x(5)\,h(5)$$
$$y(5)=1(0)+2(0)+3(1)+4(1)+0(1)+0(1)$$
$$y(5)=3+4=7$$

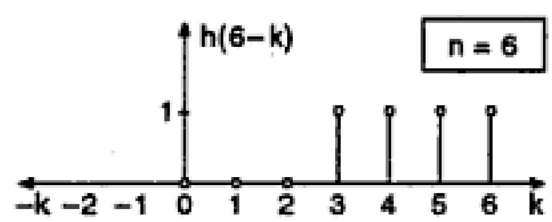


(j)

When $n=6$

Multiply the elements of Fig. (a) and Fig. (k)

$$y(6)=x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)+x(4)\,h(4)+x(5)\,h(5)+x(6)\,h(6)$$
$$y(6)=1(0)+2(0)+3(0)+4(1)+0(1)+0(1)+0(1)$$
$$y(6)=4$$

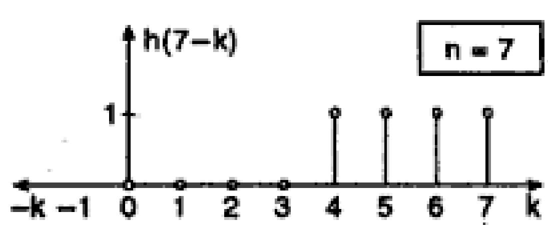


(k)

When $n=7$

Multiply the elements of Fig. (a) and Fig. (l)

$$y(7)=x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)+x(4)\,h(4)+x(5)\,h(5)+x(6)\,h(6)+x(7)\,h(7)$$
$$y(7)=1(0)+2(0)+3(0)+4(0)+0(1)+0(1)+0(1)+0(1)$$
$$y(7)=0$$



(l)

The result of the convolution is

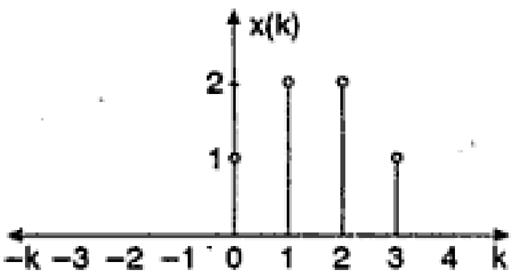
$$y(n)=x(n)*h(n)=\{1,3,6,10,9,7,4\}$$

Note If the length of $x(n)$ is n_1 and the length of $h(n)$ is n_2, then the length of the convolution sum is (n_1+n_2-1).

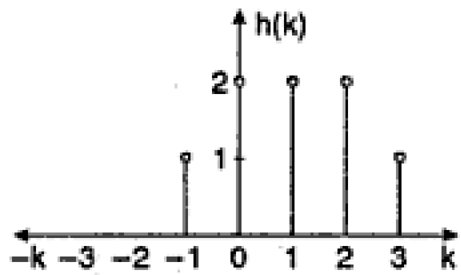
Problem 3.6 Perform the convolution of the given data sequences

$$x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}; \; h(n) = \{1, \underset{\uparrow}{2}, 2, 2, 1\}$$

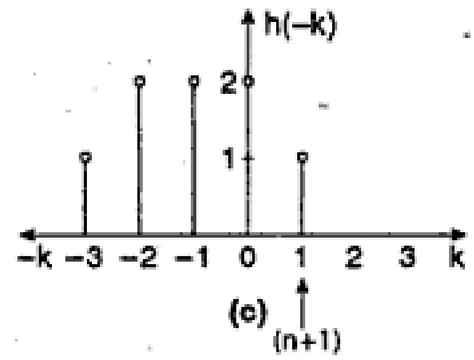
Solution



(a)



(b)



(c)

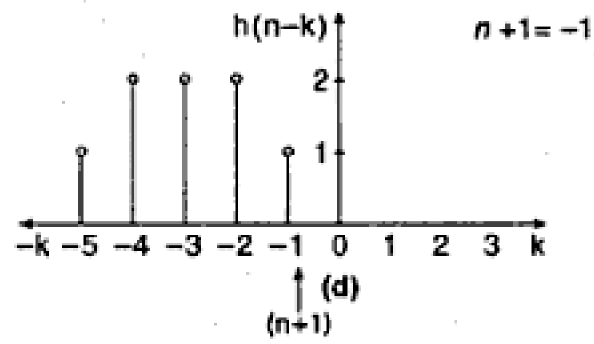
When $n+1=-1$.

Multiply the elements of Fig. (a) and Fig. (d)

$$y(-2) = x(-5)\,h(-5) + x(-4)\,h(-4) + x(-3)\,h(-3) + x(-2)\,h(-2) + x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3)$$

$$y(-2) = 0(1) + 0(2) + 0(2) + 0(2) + 0(1) + 1(0) + 2(0) + 2(0) + 1(0)$$

$$y(-2) = 0$$



(d)

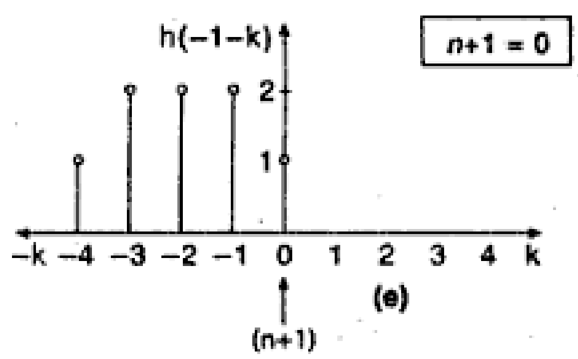
When $n+1=0$

Multiply the elements of Fig. (a) and Fig. (e)

$$y(-1) = x(-4)\,h(-4) + x(-3)\,h(-3) + x(-2)\,h(-2) + x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3)$$

$$y(-1) = 0(1) + 0(2) + 0(2) + 0(2) + 1(1) + 2(0) + 2(0) + 1(0)$$

$$y(-1) = 1$$



(e)

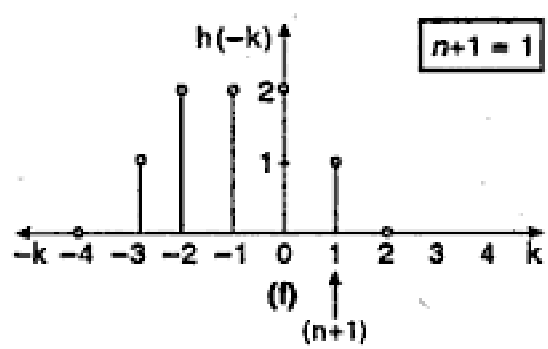
When $n+1=1$

Multiply the elements of Fig. (a) and Fig. (f)

$$y(0) = x(-3)\,h(-3) + x(-2)\,h(-2) + x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3)$$

$$y(0) = 0(1) + 0(2) + 0(2) + 1(2) + 2(1) + 2(0) + 1(0)$$

$$y(0) = 2 + 2 = 4$$



(f)

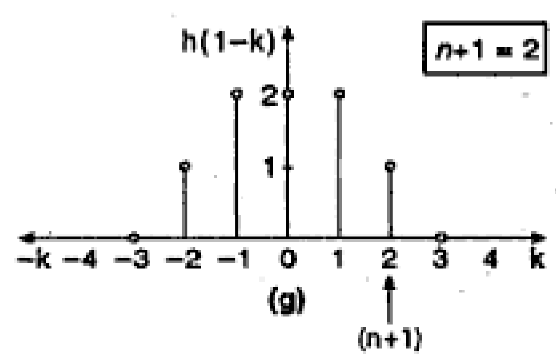
When $n+1=2$

Multiply the elements of Fig. (a) and Fig. (g)

$$y(1) = x(-2)\,h(-2) + x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3)$$

$$y(1) = 0(1) + 0(2) + 1(2) + 2(2) + 2(2) + 2(1) + 1(0)$$

$$y(1) = 2 + 4 + 2 = 8$$



(g)

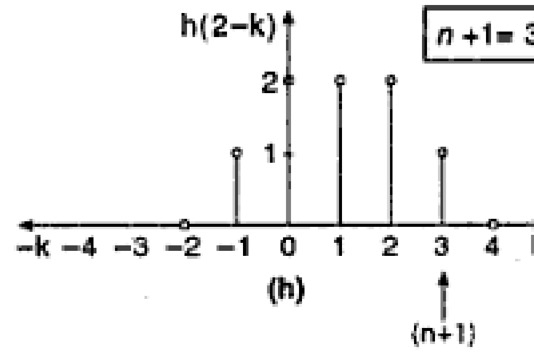
When $n+1=3$

Multiply the elements of Fig. (a) and Fig. (h)

$y(2) = x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3)$

$y(2) = 0(1) + 1(2) + 2(2) + 2(2) + 1(1)$

$y(2) = 2 + 4 + 4 + 1 = 11$



(h)

When $n+1=4$

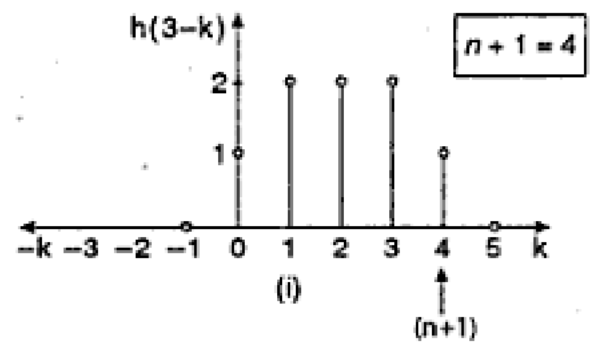
Multiply the elements of Fig. (a) and Fig. (i)

$y(3) = x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2)$

$\qquad + x(3)\,h(3) + x(4)\,h(4)$

$y(3) = 1(1) + 2(2) + 2(2) + 1(2) + 0(1)$

$y(3) = 1 + 4 + 4 + 2 = 11$



(i)

When $n+1=5$

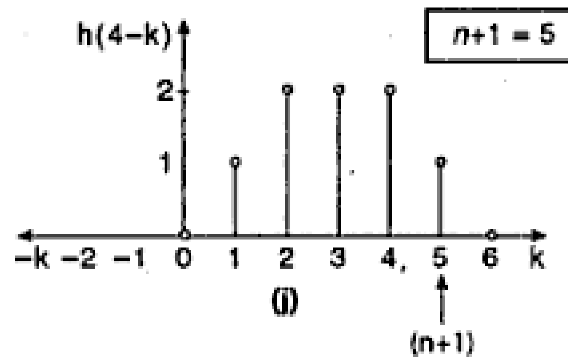
Multiply the elements of Fig. (a) and Fig. (j)

$y(4) = x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3)$

$\qquad + x(4)\,h(4) + x(5)\,h(5)$

$y(4) = 1(0) + 2(1) + 2(2) + 1(2) + 0(2) + 0(1)$

$y(4) = 2 + 4 + 2 = 8$



(j)

When $n+1=6$

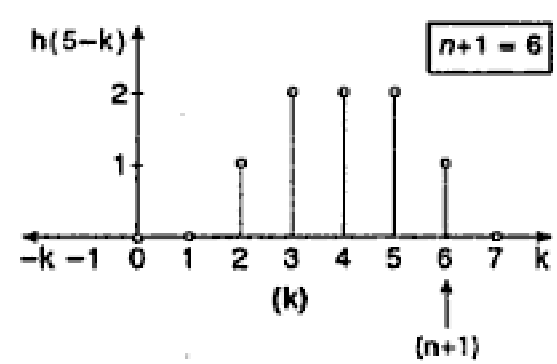
Multiply the elements of Fig. (a) and Fig. (k)

$y(5) = x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3) + x(4)\,h(4)$

$\qquad + x(5)\,h(5) + x(6)\,h(6)$

$y(5) = 1(0) + 2(0) + 2(1) + 1(2) + 0(2) + 0(2) + 1(0)$

$y(5) = 2 + 2 = 4$



(k)

When $n+1=7$

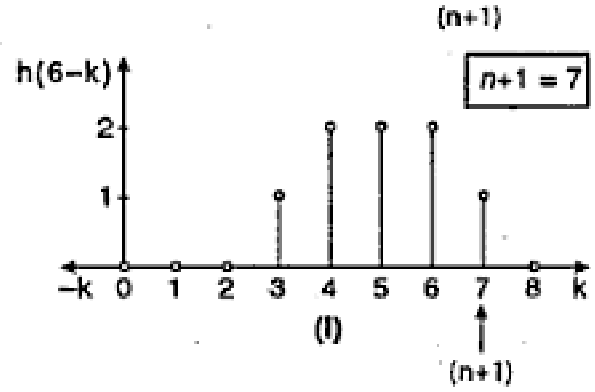
Multiply the elements of Fig. (a) and Fig. (l)

$y(6) = x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3) + x(4)\,h(4)$

$\qquad + x(5)\,h(5) + x(6)\,h(6) + x(7)\,h(7)$

$y(6) = 1(0) + 2(0) + 2(0) + 1(1) + 0(2) + 0(2) + 0(2) + 0(1)$

$y(6) = 1$



(l)

When $n+1=8$

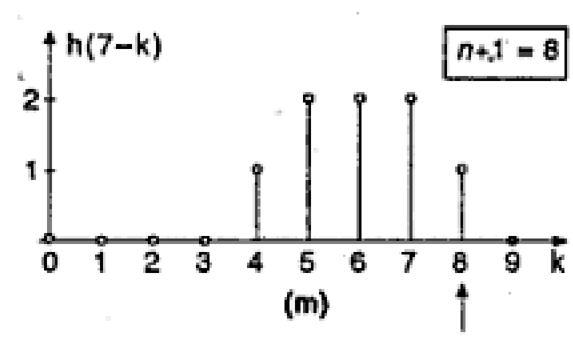
Multiply the elements of Fig. (a) and Fig. (m)

$y(7) = x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3) + x(4)\,h(4)$

$\qquad + x(5)\,h(5) + x(6)\,h(6) + x(7)\,h(7) + x(8)\,h(8)$

$y(7) = 1(0) + 2(0) + 2(0) + 1(0) + 0(1) + 0(2) + 0(2) + 0(2) + 0(1)$

$y(7) = 0$



(m)

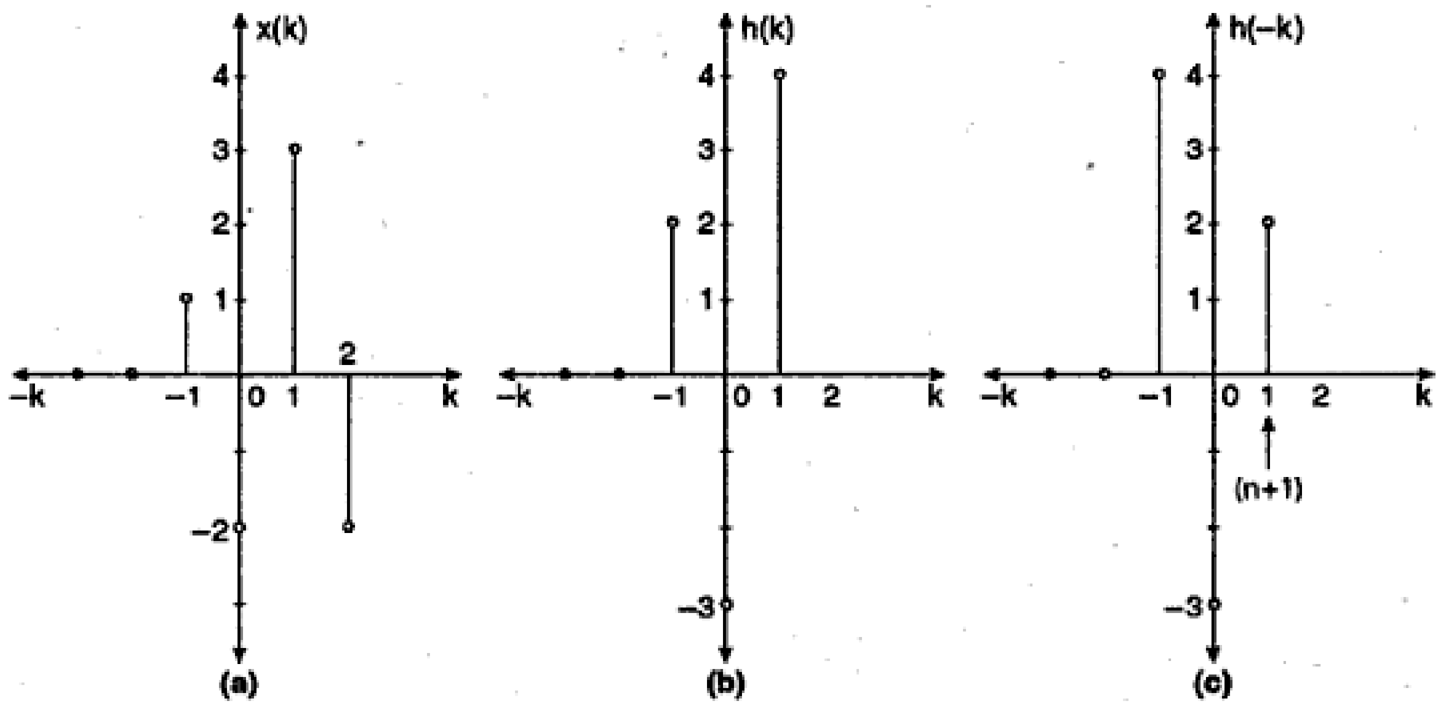
The result of the convolution is

$$y(n) = x(n) * h(n) = \{1, 4, 8, 11, 11, 8, 4, 1\}$$

Problem 3.7 Perform the convolution of the given data sequences

$$x(n) = \{1, \underset{\uparrow}{-2}, 3, -2\};\ h(n) = \{2, \underset{\uparrow}{-3}, 4\}$$

Solution



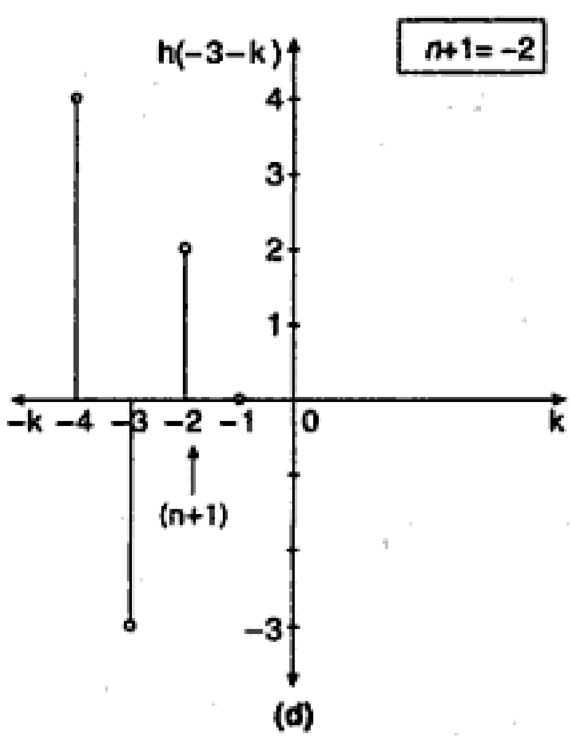
When $n = -3$ (or $n+1 = -2$)

Multiply the elements of Fig. (a) and Fig. (d)

$$y(-3) = x(-4)\,h(-4) + x(-3)\,h(-3) + x(-2)\,h(-2) + x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2)$$

$$y(-3) = 0(4) + 0(-3) + 0(2) + 1(0) + (-2)(0) + 3(0) + (-2)(0)$$

$$y(-3) = 0$$



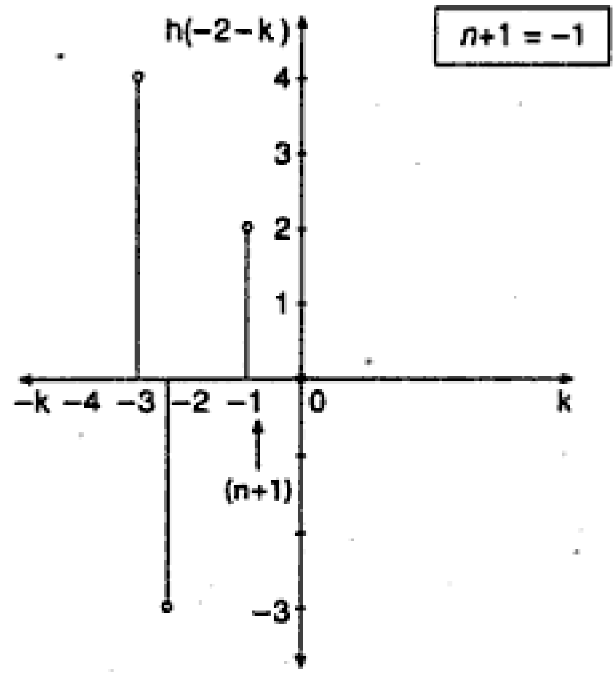
When $n = -2$ (or $n+1 = -1$)

Multiply the elements of Fig. (a) and Fig. (e)

$$y(-2) = x(-3)\,h(-3) + x(-2)\,h(-2) + x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2)$$

$$y(-2) = 0(4) + 0\,(-3) + 1(2) + (-2)\,(0) + 3(0) + (-2)\,(0)$$

$$y(-2) = 2$$



(e)

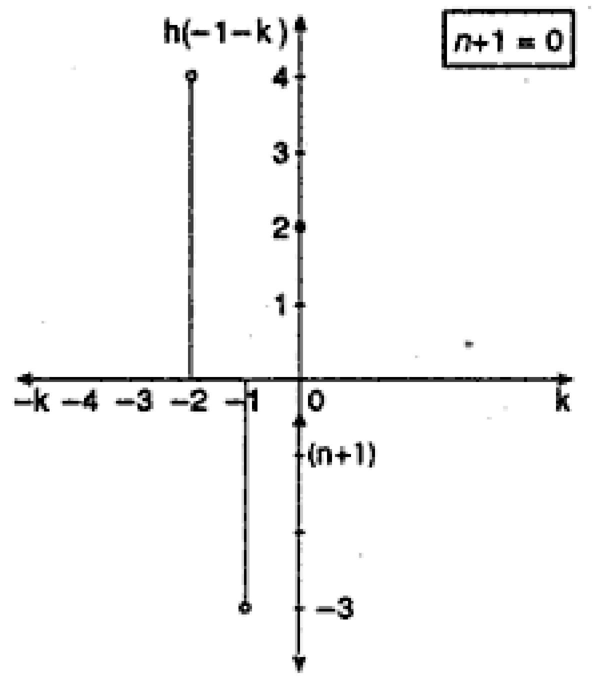
When $n = -1$ or $n+1 = 0$

Multiply the elements of Fig. (a) and Fig. (f)

$$y(-1) = x(-2)\,h(-2) + x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2)$$

$$y(-1) = 0(4) + 1\,(-3) + (-2)\,(2) + 3(0) + (-2)\,(0)$$

$$y(-1) = -3 - 4 = -7$$



(f)

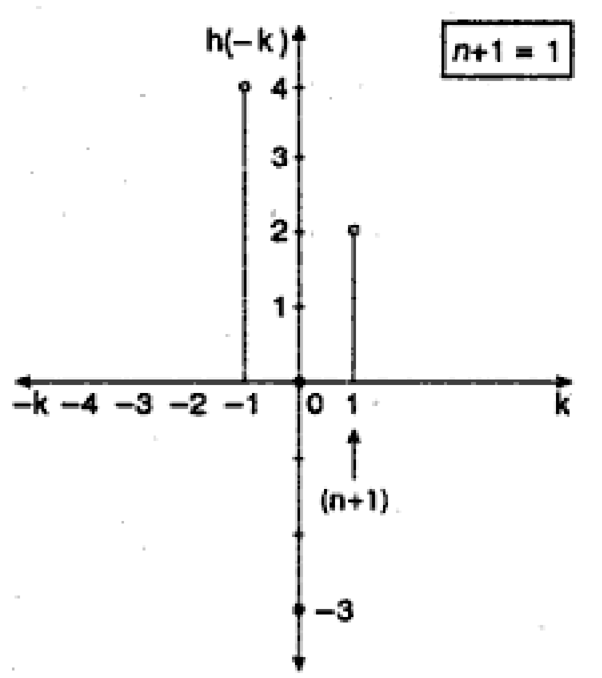
When $n = 0$ (or $n+1 = 1$)

Multiply the elements of Fig. (a) and Fig. (g)

$$y(0) = x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2)$$

$$y(0) = 1(4) + (-2)\,(-3) + 3(2) + (-2)\,0$$

$$y(0) = 4 + 6 + 6 = 16$$



(g)

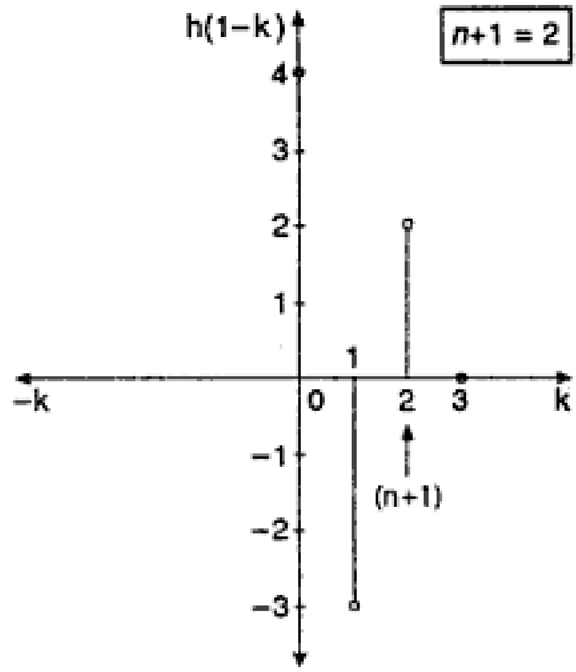
When $n = 1$ (or $n + 1 = 2$)

Multiply the elements of Fig. (a) and Fig. (h)

$y(1) = x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2)$

$y(1) = 1(0) + (-2)(4) + 3(-3) + (-2)(2)$

$y(1) = -8 - 9 - 4 = -21$



(h)

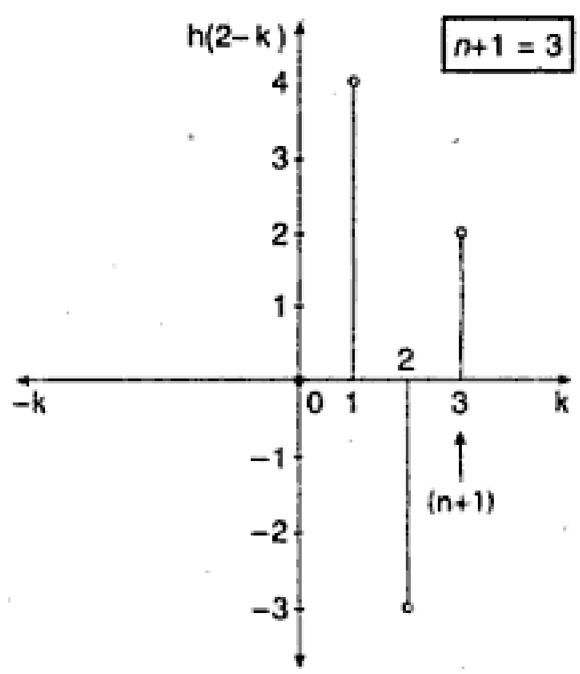
When $n = 2$ (or $n + 1 = 3$)

Multiply the elements of Fig. (a) and Fig. (i)

$y(2) = x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3)$

$y(2) = 1(0) + (-2)(0) + 3(4) + (-2)(-3) + 0(2)$

$y(2) = 12 + 6 = 18$



(i)

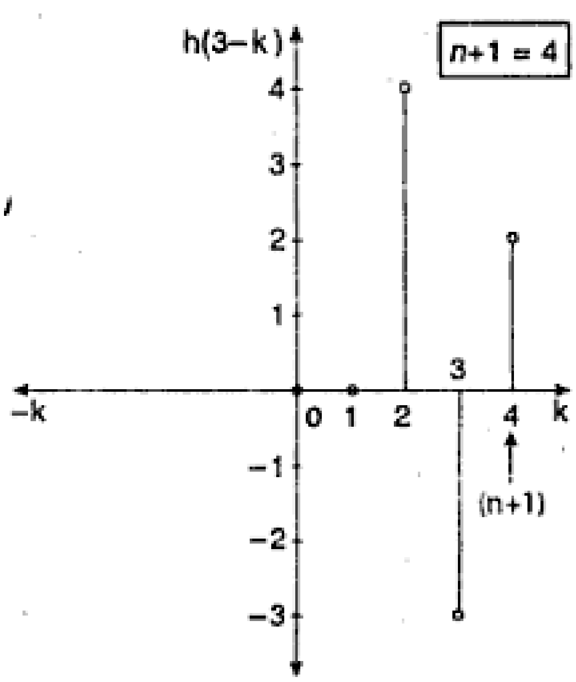
When $n = 3$ (or $n + 1 = 4$)

Multiply the elements of Fig. (a) and Fig. (j)

$y(3) = x(-1)\,h(-1) + x(0)\,h(0) + x(1)\,h(1) + x(2)\,h(2) + x(3)\,h(3) + x(4)$

$y(3) = 1(0) + (-2)(0) + 3(0) + (-2)(4) + 0(-3) + 0(2)$

$y(3) = -8$



(j)

When $n=4$ (or $n+1=5$)

Multiply the elements of Fig. (a) and Fig. (k)

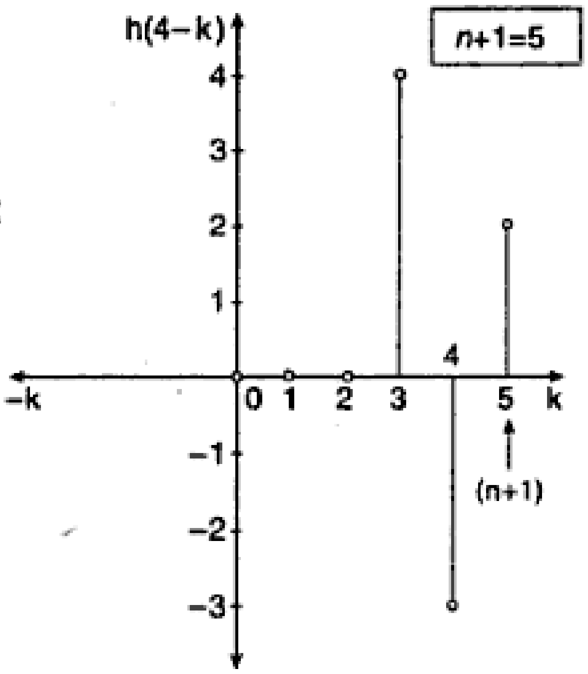
$y(4)=x(-1)\,h(-1)+x(0)\,h(0)+x(1)\,h(1)+x(2)\,h(2)+x(3)\,h(3)+x($

$y(4)=1(0)+(-2)0+3(0)+(-2)0+0(4)+0(-3)+0(2)$

$y(4)=0$

The result of the convolution is

$$y(n)=x(n)*h(n)=\{2,-7,16,-21,18,-8\}$$



(k)

■ 3.5 LINEAR CONVOLUTION USING CROSS-TABLE METHOD

Let us consider the convolution,

$$y(n)=x(n)*h(n)$$

where $x(n)=\{x_1(n), x_2(n), x_3(n),\ldots\}$ is the input signal and $h(n)=\{h_1(n), h_2(n), h_3(n),\ldots,\}$ is the impulse response.

The convolution of $x(n)$ and $h(n)$ can be performed as

	$x_1(n)$	$x_2(n)$	$x_3(n)$
$h_1(n)$	$x_1(n)\,h_1(n)$	$x_2(n)\,h_1(n)$	$x_3(n)\,h_1(n)$
$h_2(n)$	$x_1(n)\,h_2(n)$	$x_2(n)\,h_2(n)$	$x_3(n)\,h_2(n)$
$h_3(n)$	$x_1(n)\,h_3(n)$	$x_2(n)\,h_3(n)$	$x_3(n)\,h_3(n)$

Fig. 3.7 Convolution of x(n) and h(n)

Procedure

1. Multiply each row element with column element
2. Draw a diagonal line as shown in the Fig. 3.7.
3. Add the diagonal terms

   $y(n)=\{x_1(n)\,h_1(n), \{(x_1(n)\,h_2(n)+x_2(n)\,h_1(n)\}, \{(x_1(n)\,h_3(n)+x_2(n)\,h_2(n)+x_3(n)\,h_1(n)\},\ldots\}$

SOLVED PROBLEMS

Problem 3.8 Perform the convolution of $x(n)$ and $h(n)$, where $x(n)=\{1,2,3,4\}$ and $h(n)=\{1,1,1,1\}$.

Solution

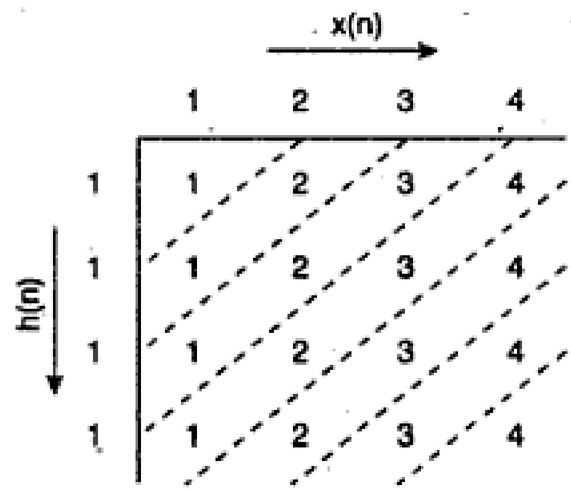


Fig. 3.8

$$y(n)=x(n)*h(n)$$
$$y(n)=\{1,(1+2),(1+2+3),(1+2+3+4),(2+3+4),(3+4),4\}$$
$$y(n)=\{1,\ 3,\ 6,\ 10,\ 9,\ 7,\ 4\}$$

Problem 3.9 Perform the convolution of $x(n)$ and $h(n)$. $x(n)=\{1,-2,3,-4\}$, $h(n)=\{4,-3,2,-1\}$

Solution

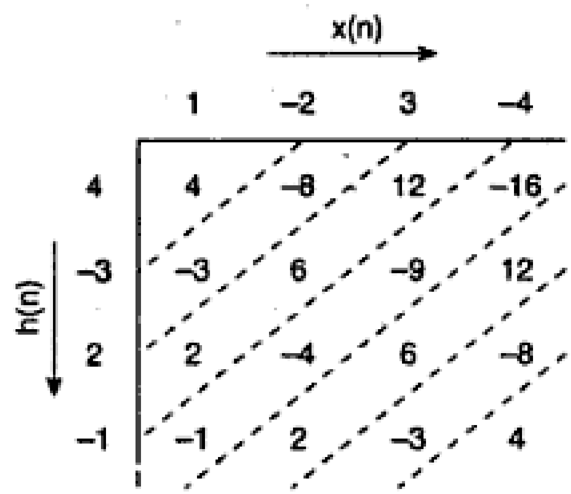


Fig. 3.9

$$y(n)=x(n)*h(n)$$
$$y(n)=\{4,-11,20,-30,20,-11,4\}$$

■ 3.6 LINEAR CONVOLUTION USING MATRIX METHOD

In this method, the data sequences are represented as a matrix. If the length of signal $x(n)$ is N_1 and the length of the impulse response $h(n)$ is N_2, then the matrix X can be obtained from $x(n)$, whose order will be $(N_1+N_2-1)\times N_1$ and the matrix H can be obtained from $h(n)$, whose order will be $N_2\times 1$ such that $Y=XH$.

Let us understand the convolution using matrix method with the following examples.

SOLVED PROBLEMS

Problem 3.10 Find $y(n) = x(n) * h(n)$ using the matrix method. $x(n) = \{1, 2, 3, 4\}$; $h(n) = \{1, 1, 1, 1\}$

Solution

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Y = XH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+1 \\ 3+2+1 \\ 4+3+2+1 \\ 4+3+2 \\ 4+3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \\ 9 \\ 7 \\ 4 \end{bmatrix}$$

Problem 3.11 Find the convolution of the following data sequences using the matrix method. $x(n) = \{1, -2, 3, -4\}$; $h(n) = \{4, -3, 2, -1\}$

Solution

$$Y = XH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -4 & 3 & -2 & 1 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ 20 \\ -30 \\ 20 \\ -11 \\ 4 \end{bmatrix}$$

Problem 3.12 The impulse response of an LTI system is $h(n) = \left(\frac{1}{3}\right)^n u(n)$. Determine the output of the system $y(n)$ at

(i) $n = -2$ (ii) $n = 2$ and (iii) $n = +4$, when input signal $x(n) = u(n)$.

Solution By convolution sum,

$$y(n) = h(n) * x(n) = x(n) * h(n) \quad \text{(Commutative property)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(n) h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} u(n)\left[\left(\frac{1}{3}\right)^{n-k} u(n-k)\right]$$

$$y(n) = \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{n-k}$$

For $n = -2$ $\quad y(-2) = 0$

For $n = 2$

$$y(2) = \sum_{k=0}^{2}\left(\frac{1}{3}\right)^{2-k} = \frac{1}{9}\sum_{k=0}^{2} 3^k$$

Hint $\sum_{k=0}^{N} \alpha^n = \frac{1-\alpha^{N+1}}{1-\alpha}$

$$y(2) = \frac{1}{9}\left[\frac{1-(3)^3}{1-3}\right] = \frac{13}{9}$$

For $n = 4$

$$y(4) = \sum_{k=0}^{4}\left(\frac{1}{3}\right)^{4-k}$$

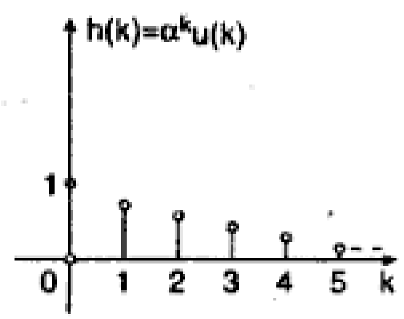
$$y(4) = \left(\frac{1}{3}\right)^4 \sum_{k=0}^{4} 3^k = \frac{1}{81}\left(\frac{1-3^5}{1-3}\right) = \frac{121}{81}$$

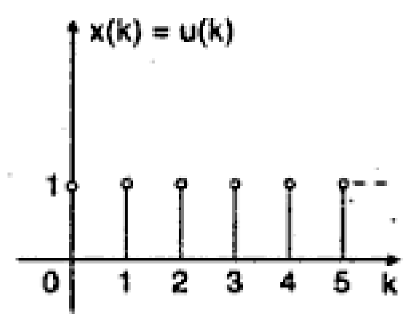
Problem 3.13 A LTI system has the impulse response $h(n) = \alpha^n u(n)$, $|\alpha| < 1$. Determine the output of the system when the input $x(n) = u(n)$.

Solution

$$y(n) = x(n) * h(n) = h(n) * x(n) \quad \text{(Commutative property)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

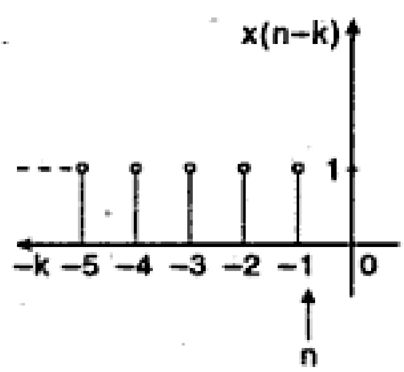




When $n < 0$

$$x(n-k) = \begin{cases} u(n-k), & k < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$h(k) = \begin{cases} \alpha^k, & k \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



Since $x(n-k)$ and $h(k)$ has no overlap terms, therefore output is zero, that is,

$$y(n) = 0, \qquad n < 0$$

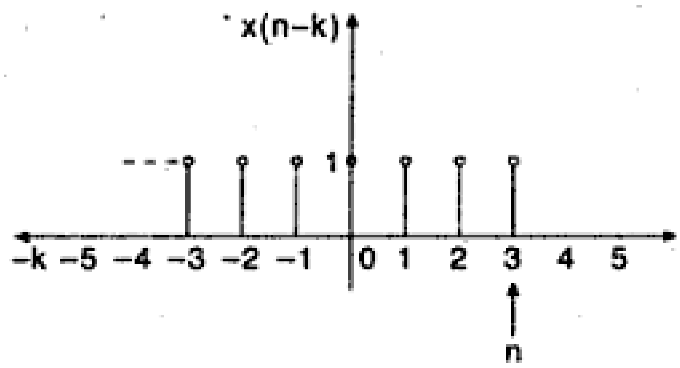
When $n > 0$

$$y(n) = \sum_{k=0}^{+n} h(k)\,x(n-k)$$

$$y(n) = \sum_{k=0}^{n} \alpha^{k}.1$$

$$y(n) = \left[\frac{1-(\alpha)^{n+1}}{1-\alpha}\right]$$

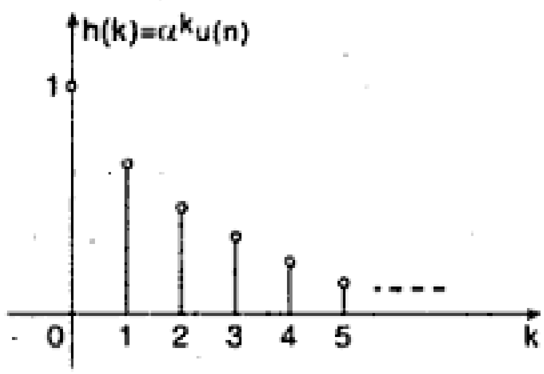
$$y(n) = \begin{cases} 0, & n < 0 \\ \left[\dfrac{1-(\alpha)^{n+1}}{1-\alpha}\right], & n \geq 0 \end{cases}$$

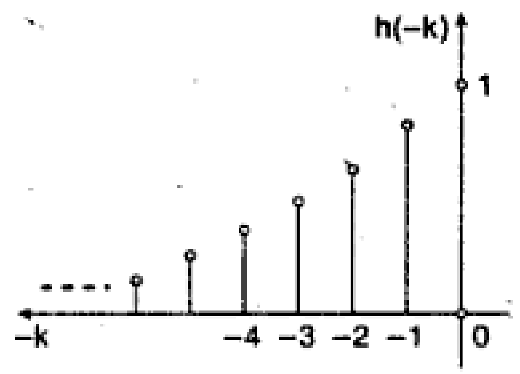


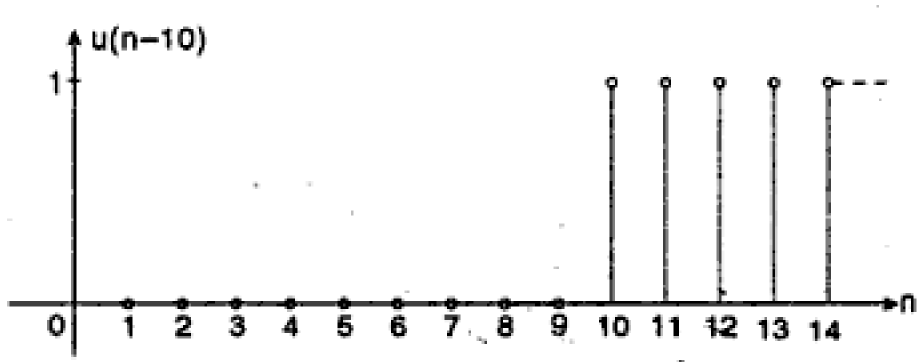
The lower value of x $(n\text{-}k)$ is still exist in the negative x-axis, where as the value of h (k) is zero in these region. Hence the lower value of summation takes the value 0. The upper value of summation is taken as variable n due to the fact that n can take any value from 0 to ∞.

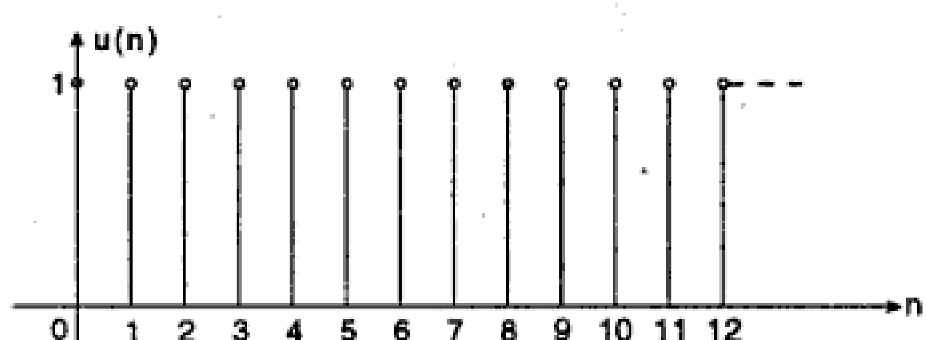
Problem 3.14 An LTI system has the impulse response $h(n) = \alpha^{n}u(n)$, $0 < \alpha < 1$. Determine the output of the system when the input $x(n) = u(n) - u(n-10)$.

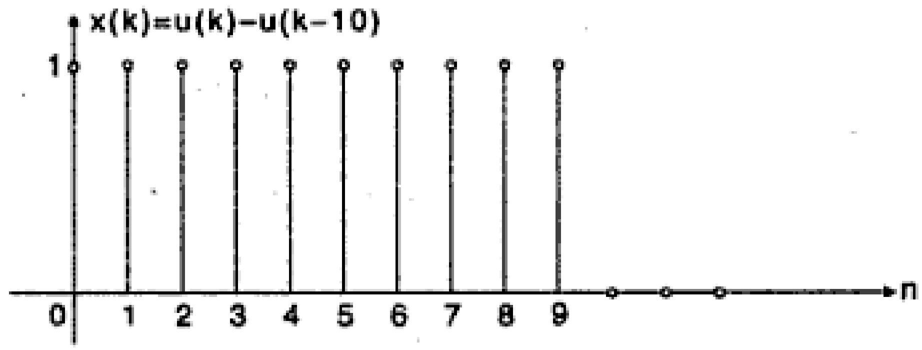
Solution

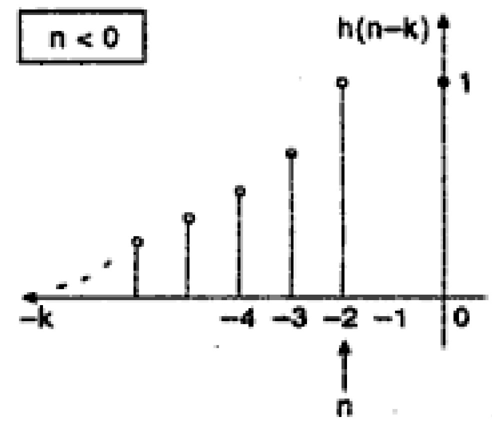












When $n < 0$

$$x(k) = \begin{cases} u(k), & 9 \ge k \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

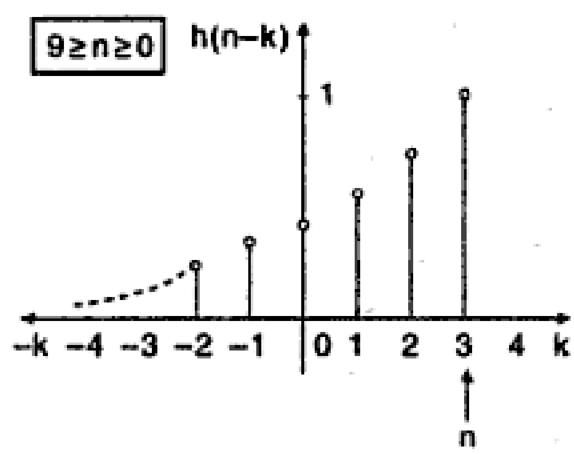
$$h(n-k) = \begin{cases} \alpha^{(n-k)}, & k \le n \\ 0, & \text{otherwise} \end{cases}$$

Since $x(k)$ and $h(n-k)$ has no overlap terms, therefore output response is zero.

$$y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k) = 0 \qquad n < 0$$

When $9 \ge n \ge 0$ [range of $h(n-k)$ is compared with $x(k)$]

$$y(n) = \sum_{k=0}^{n} x(k)\,h(n-k)$$



$$y(n) = \sum_{k=0}^{n} \alpha^{n-k} = \alpha^{n} \sum_{k=0}^{n} \alpha^{-k}$$

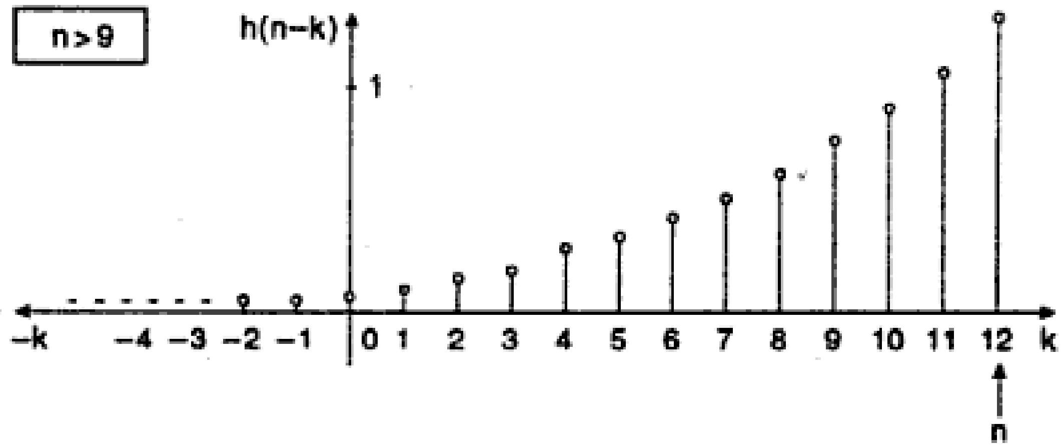
$$y(n) = \alpha^{n} \sum_{k=0}^{n} \left(\frac{1}{\alpha}\right)^{k} = \alpha^{n} \left[\frac{1-\left(\frac{1}{\alpha}\right)^{n+1}}{1-\left(\frac{1}{\alpha}\right)} \right]$$

Hint $\displaystyle\sum_{n=0}^{N} a^{n} = \frac{1-a^{N+1}}{1-a}$

$$y(n) = \frac{\alpha^{n+1}-1}{\alpha-1},\ 9 \ge n > 0$$

The lowest value of $h(n-k)$ still exists in the negative x-axis, where the value of $x(k)$ is zero. Hence the lower value of summation takes the value zero. The upper value of $h(n-k)$ can take any variable 'n' between 0 to 9, hence it is simply n. Hence the upper value of summation takes the variable value n.

When $n > 9$



$$y(n) = \sum_{k=0}^{9} x(k)h(n-k)$$

The lowest value of $h(n–k)$ exists in the negative x-axis, where the value of $x(k)$ is zero. Hence the lower value of summation takes zero. The upper range of $h(n–k)$ has exceeded the maximum range of $x(k)$, (i.e. 9), hence $h(n–k)$ becomes zero for $n > 9$. Therefore, the upper value of $h(n–k)$ is limited to 9.

$$y(n) = \sum_{k=0}^{9} \alpha^{n-k}$$

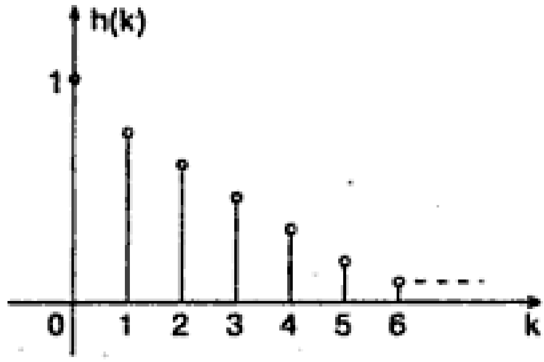
$$y(n) = \alpha^n \sum_{k=0}^{9} (1/\alpha)^k$$

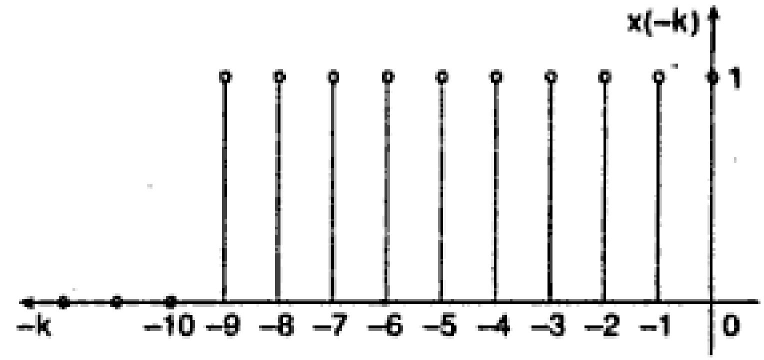
$$y(n) = \alpha^k \left[\frac{1-(1/\alpha)^{10}}{1-(1/\alpha)}\right] \quad n > 9$$

$$y(n) = \begin{cases} 0, & n < 0 \\ \alpha^n \left[\dfrac{1-(1/\alpha)^{n+1}}{1-(1/\alpha)}\right], & 9 \geq n \geq 0 \\ \alpha^n \left[\dfrac{1-(1/\alpha)^{10}}{1-(1/\alpha)}\right], & n > 9 \end{cases}$$

Problem 3.15 Repeat problem 3.13 by interchanging the position of $x(n)$ and $h(n)$.

Solution



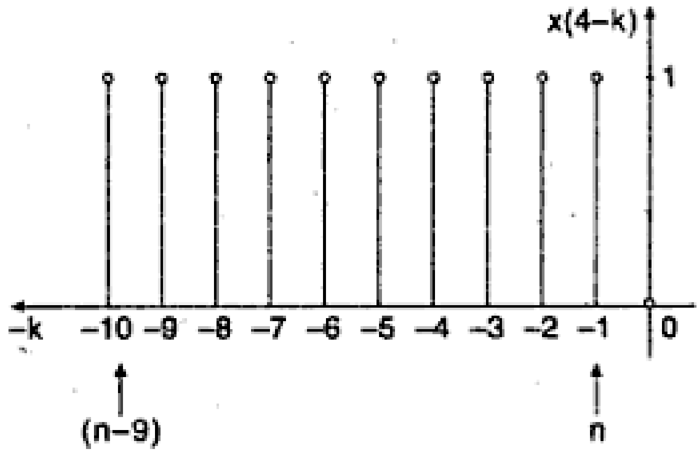


When $n < 0$

$$h(k) = \begin{cases} \alpha^n u(n), & k \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x(n-k) = \begin{cases} u(n-k), & k \leq n \\ 0, & \text{otherwise} \end{cases}$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = 0$$

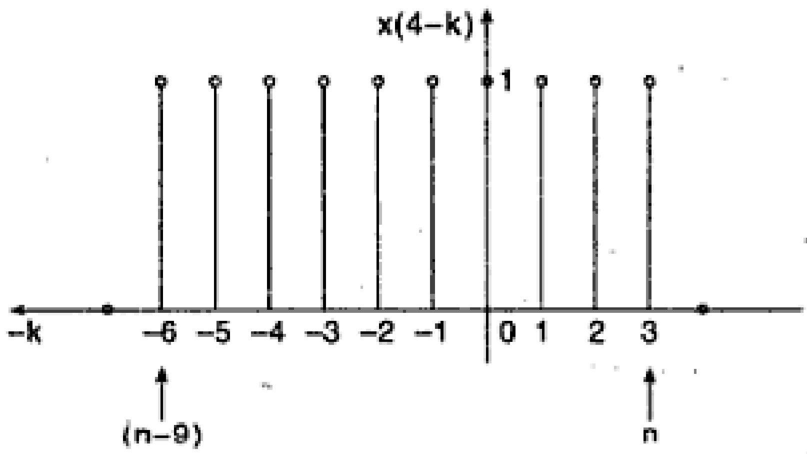


Both $h(n)$ and $x(n)$ have no overlap terms. Therefore output response is zero.

When $9 > n \geq 0$

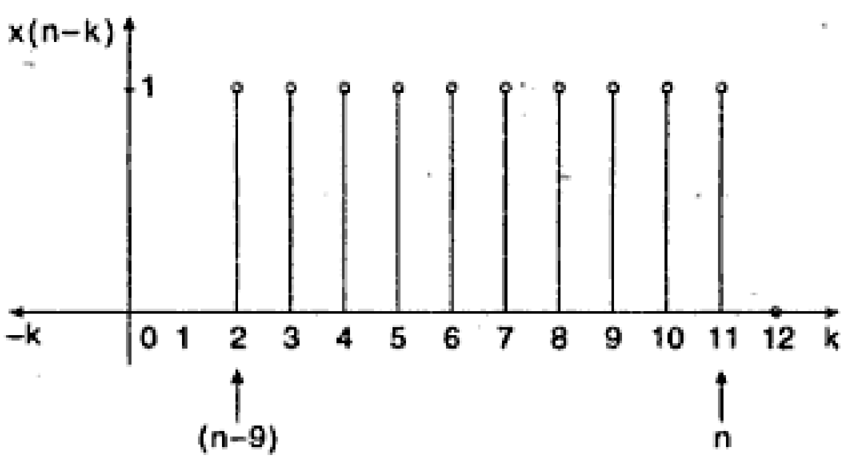
$$y(n) = \sum_{k=0}^{n} h(k)x(n-k)$$

$$y(n) = \sum_{k=0}^{n} a^k = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) \qquad 9 > n \geq 0$$



The lowest value exists in the negative x-axis, where the value of $x(k)$ is zero. Hence, the lower value of summation takes 0. The upper range of $x(h-k)$ is well below the maximum range of $h(k)$, hence the upper value of summation takes the variable value n.

When $n \geq 9$



In this case, the lower range of $x(n-k)$ has exceeded the lower range of $h(k)$, hence the lower range of summation must be $(n-9)$. The upper range of $x(n-k)$ is less than the maximum range of $h(k)$ (i.e. ∞), the upper range of summation must be n itself.

$$y(n) = \sum_{k=n-9}^{n} h(n)x(n-k) = \sum_{k=n-9}^{n} \alpha^k$$

Let $k - n + 9 = 0 = m$

$$y(n) = \sum_{m=0}^{9} \alpha^{(m+n-9)}$$

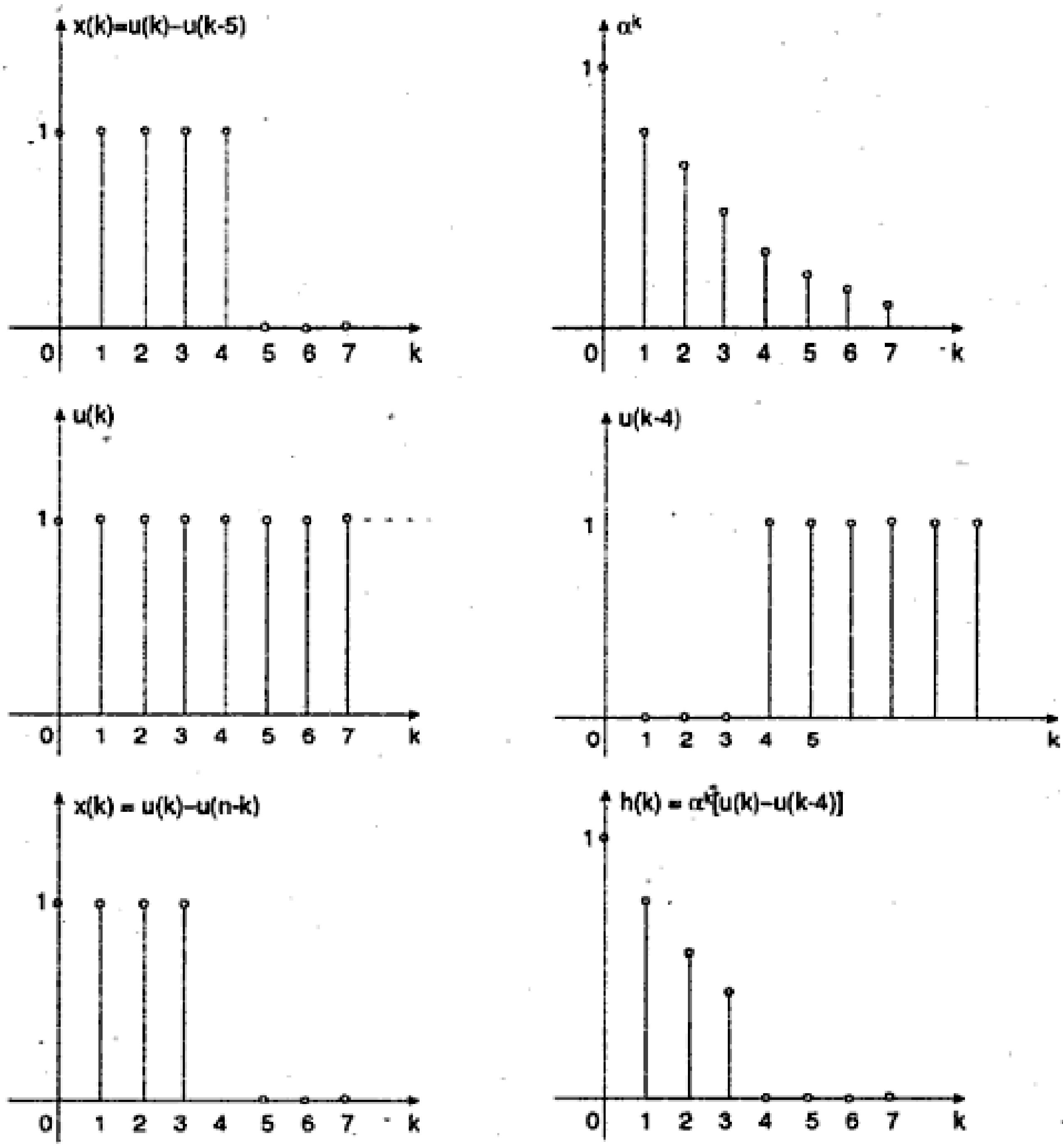
$$y(n) = \alpha^{k-9} \sum_{m=0}^{9} \alpha^{m}$$

$$y(n) = \alpha^{n-9} \left(\frac{1-\alpha^{10}}{1-\alpha} \right) \quad n \geq 9$$

$$y(n) = \begin{cases} 0, & n < 0 \\ \left(\dfrac{1-\alpha^{n+1}}{1-\alpha} \right), & 9 \geq n \geq 0 \\ \alpha^{n-9} \left(\dfrac{1-\alpha^{10}}{1-\alpha} \right), & n \geq 9 \end{cases}$$

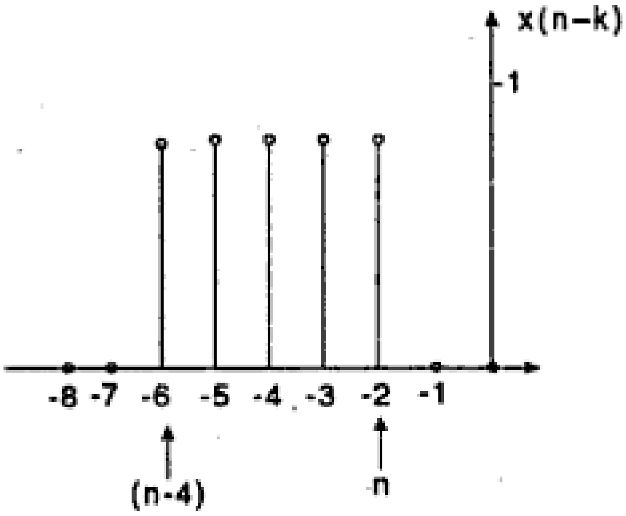
Problem 3.16 An LTI system has the impulse response $h(n) = \alpha^n [u(n) - u(n-4)]$ $0 < \alpha < 1$. Determine the output of the system when the input $x(n) = [u(n) - u(n-5)]$.

Solution



When $n < 0$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = 0$$



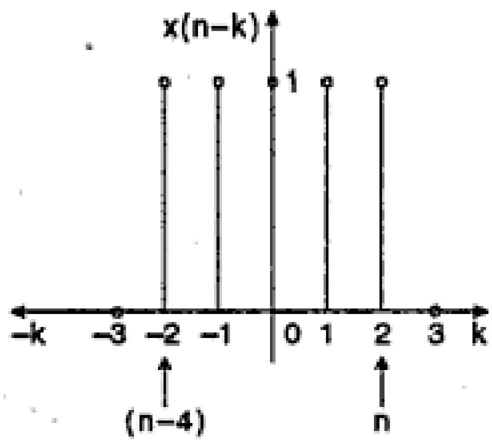
Both $h(k)$ and $x(n-k)$ are not overlapped, therefore, output response is zero.

When $3 \geq n \geq 0$

$$y(n) = \sum_{n=0}^{n} h(k)x(n-k)$$

$$y(n) = \sum_{k=0}^{n} \alpha^{k}$$

$$y(n) = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right), 3 > n \geq 0$$



The lower range of $x(n-k)$ exists in the negative x-axis, where the value of $h(k)$ does not exist. Therefore, the lower value of summation is zero. The upper range of $x(n-k)$ varies in the range $3 \geq n \geq 0$ of $h(k)$, hence the upper value of summation is a variable n.

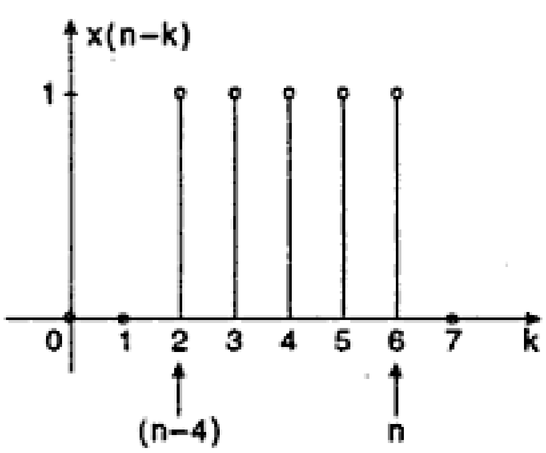
When $7 \geq n \geq 3$

$$y(n) = \sum_{k=(n-4)}^{3} h(k)x(n-k)$$

$$y(n) = \sum_{k=n-4}^{3} \alpha^{k}$$

Let $k - n + 4 = 0 = m$ $\quad y(n) = \sum_{m=0}^{7-n} \alpha^{m+n-4} = \alpha^{n-4} \sum_{m=0}^{7-n} \alpha^{m}$

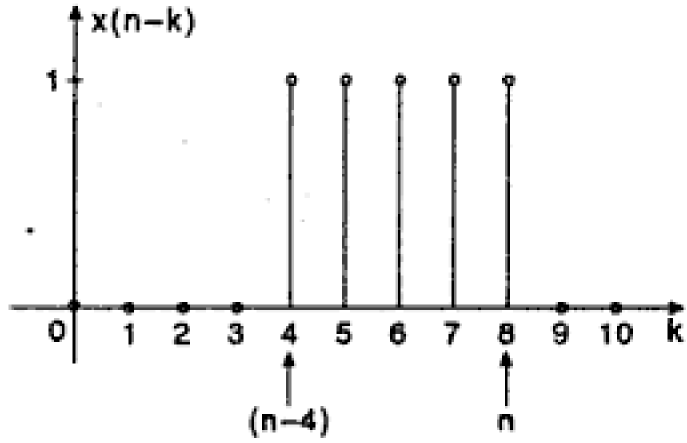
$$y(n) = \alpha^{n-4}\left(\frac{1-\alpha^{8-n}}{1-\alpha}\right) \quad 7 \geq n \geq 3$$



The lower range of $x(n-k)$ has exceeded the lower range of $h(k)$. Therefore, the lower value of summation is $(n-4)$. The upper range of $x(n-k)$ crossed the maximum upper value of $x(k)$, ie., 3 , hence the upper value of summation is 3.

When $n \geq 8$

$$y(n) = \sum_{k=4}^{4} h(k)x(n-4) = 0, \quad n \geq 8$$

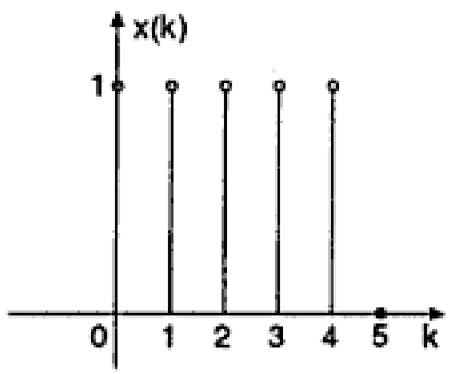


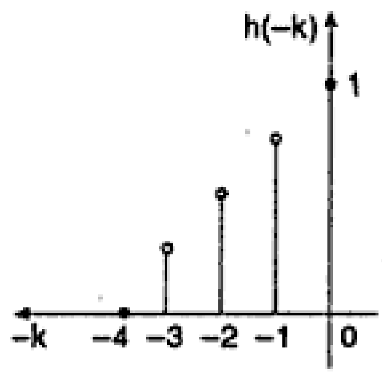
$$y(n) = \begin{cases} 0, & n < 0 \\ \left(\dfrac{1-\alpha^{n+1}}{1-\alpha}\right), & 3 \geq n \geq 0 \\ \alpha^{n-4}\left(\dfrac{1-\alpha^{8-n}}{1-\alpha}\right), & 7 \geq n \geq 3 \\ 0, & n > 8 \end{cases}$$

In this condition, both $h(k)$ and $x(n-k)$ are not overlapped. Hence, output response is zero.

Problem 3.17 Repeat problem 3.15 by interchanging $x(k)$ and $h(k)$.

Solution

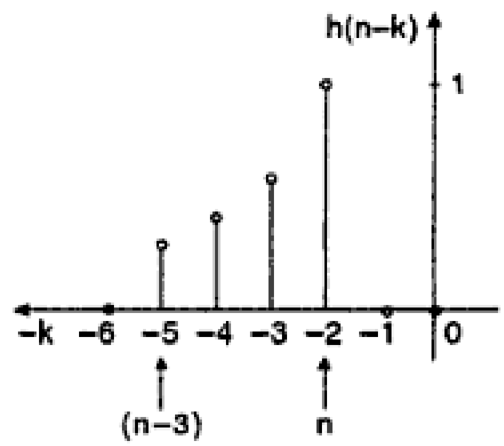




When $n < 0$

$$x(k) = \begin{cases} u(k), & 4 \geq k \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$h(n-k) = \begin{cases} \alpha^{(n-k)}, & k < n \\ 0, & \text{otherwise} \end{cases}$$



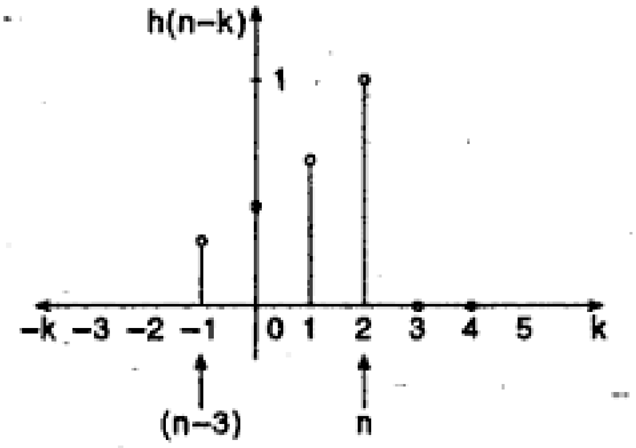
Both $x(k)$ and $h(n-k)$ are not overlapped. Therefore output response is zero.

Therefore, $$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k) = 0$$

When $3 \geq n \geq 0$

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k) = \sum_{k=0}^{n} \alpha^{n-k}$$

$$y(n) = \alpha^n \sum_{k=0}^{n} \left(\frac{1}{\alpha}\right)^k = \alpha^n \left[\frac{1-\left(\frac{1}{\alpha}\right)^{n+1}}{1-\left(\frac{1}{\alpha}\right)}\right] \quad 3 \geq n \geq 0$$



The lower range of $h(n-k)$ is in the negative x-axis, where $x(k)$ has no value. Therefore, the lower value of the summation is 0. The upper range of $h(n-k)$ is above 0 but less than $n = 4$, hence the value of the summation is a variable n.

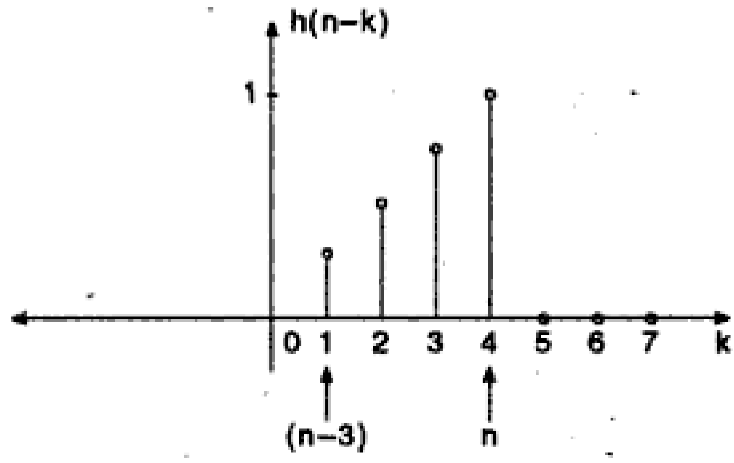
When $4 \geq n \geq 3$

$$y(n) = \sum_{k=n-3}^{n} x(k)h(n-k) = \sum_{k=n-3}^{n} \alpha^{n-k}$$

Let $k-n+3=0$

$$y(n) = \sum_{m=0}^{3} \alpha^{3-m} = \alpha^3 \sum_{m=0}^{3} \alpha^{-m}$$

$$y(n) = \alpha^3 \left[\frac{1-\left(\frac{1}{\alpha}\right)^4}{1-\left(\frac{1}{\alpha}\right)}\right] \quad 4 \geq n \geq 3$$



The lower range of $h(n-k)$ has exceeded the lower range of $x(k)$. Therefore, the lower value of the summation is $(n-3)$. The upper range of $h(n-k)$ varies in the range $4 \geq n \geq 3$, hence the upper value of the summation is a varible n.

When $7 \geq n \geq 4$

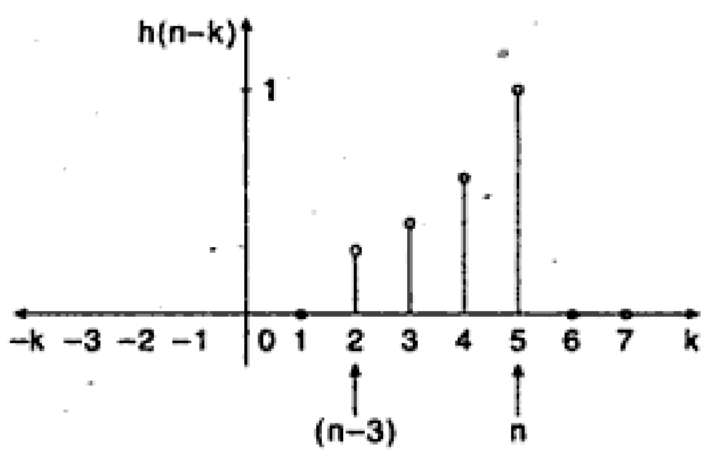
$$y(n) = \sum_{k=(n-3)}^{4} x(k)h(n-k) = \sum_{k=(n-3)}^{4} \alpha^{n-k}$$

$$y(n) = \sum_{m=0}^{7-n} \alpha^{3-m}$$

Let $k-n+3=0$

$$y(n) = \sum_{m=0}^{7-n} \alpha^{3-m}$$

$$y(n) = \alpha^3 \sum_{m=0}^{7-n} \left(\frac{1}{\alpha}\right)^m = \alpha^3 \left[\frac{1-\left(\frac{1}{\alpha}\right)^{8-n}}{1-\left(\frac{1}{\alpha}\right)}\right] \quad 7 \geq n \geq 4$$

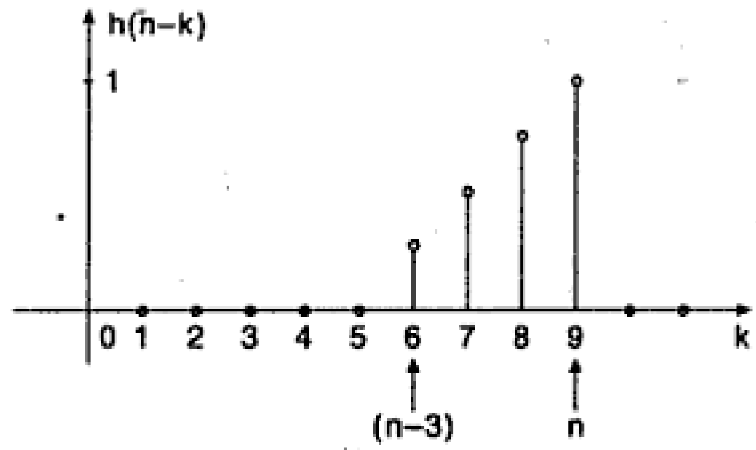


The lower range of $h(n-k)$ has exceeded the lower range of $x(k)$. Therefore, the lower value of the summation is $(n-3)$. The upper range of $h(n-k)$ varies in the range $7 \geq n \geq 4$. Since the upper range of $h(n-k)$ exceeds the upper range of $x(k)$, the upper value of the summation is 4.

When $n \geq 8$

$$y(n) = 0 \quad n \geq 8$$

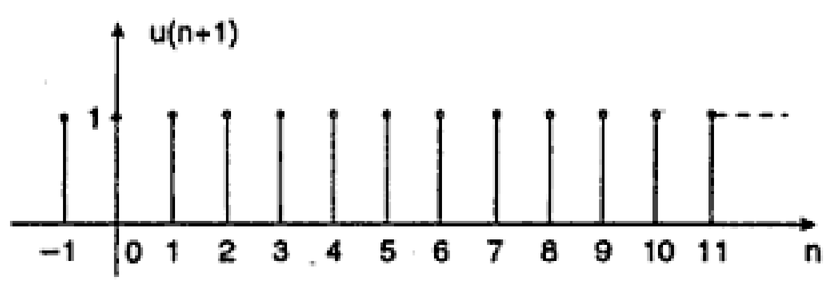
$$y(n) = \begin{cases} 0 & n < 0 \\ \alpha^n \left[\dfrac{1-\left(\dfrac{1}{\alpha}\right)^{n+1}}{1-\left(\dfrac{1}{\alpha}\right)} \right] & 3 \geq n \geq 0 \\ \alpha^3 \left[\dfrac{1-\left(\dfrac{1}{\alpha}\right)^{4}}{1-\left(\dfrac{1}{\alpha}\right)} \right] & 4 \geq n \geq 3 \\ \alpha^3 \left[\dfrac{1-\left(\dfrac{1}{\alpha}\right)^{8-n}}{1-\left(\dfrac{1}{\alpha}\right)} \right] & 7 \geq n \geq 4 \\ 0 & n \geq 8 \end{cases}$$

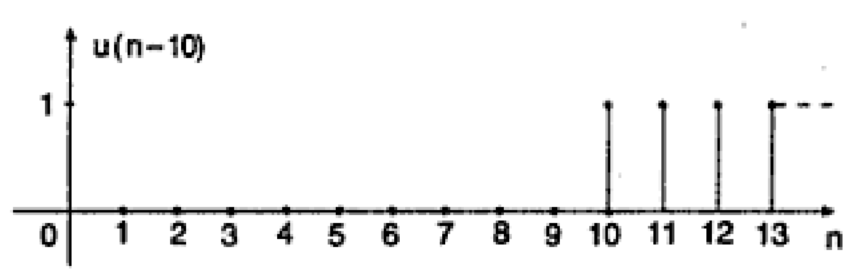


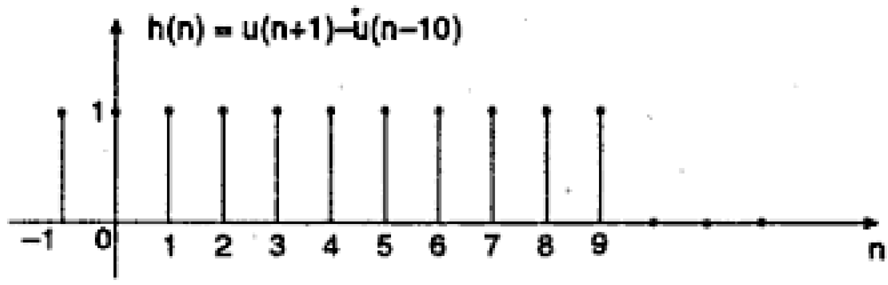
Both $x(k)$ and $h(n-k)$ are not overlapped. Therefore output response is zero.

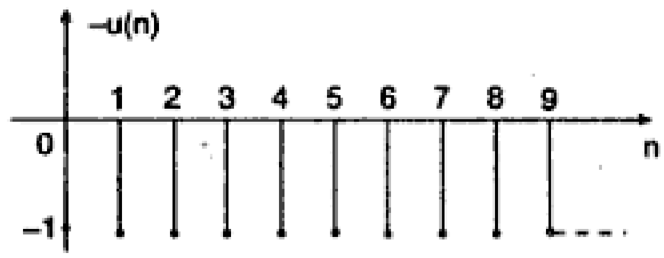
Problem 3.18 An LTI system has an impulse response $h(n) = u(n+1) - u(n-10)$. Determine the output of the system when the input $x(n) = -u(n) + 2u(n-3) - u(n-6)$.

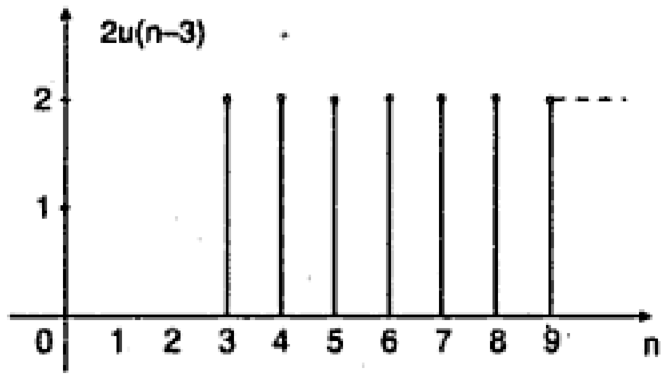
Solution

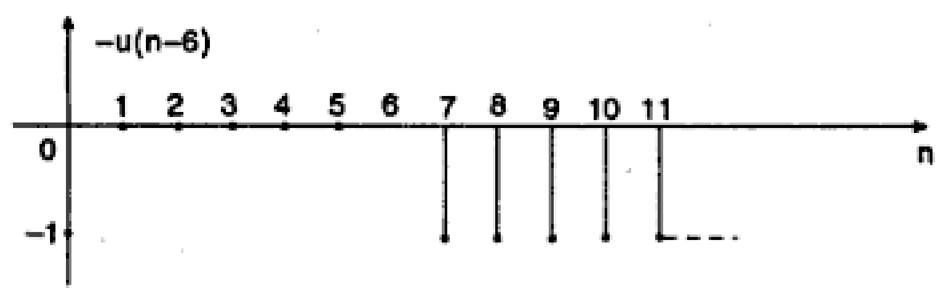


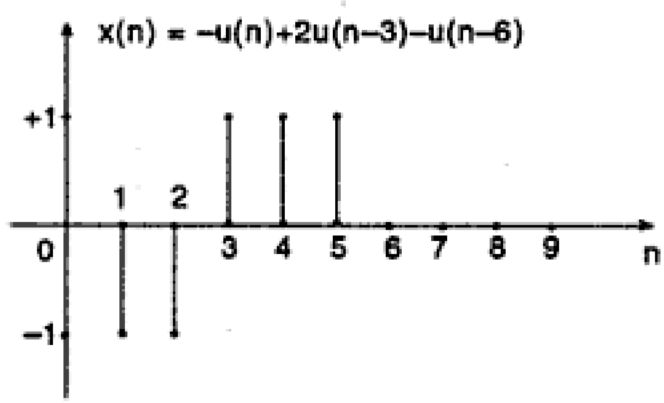


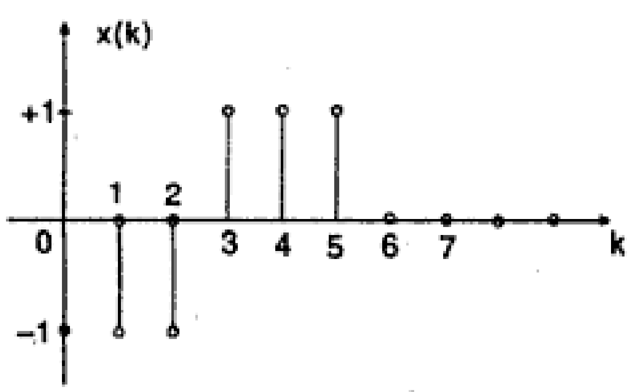

h(n) = u(n+1)–u(n–10)
1
–1 0 1 2 3 4 5 6 7 8 9
n

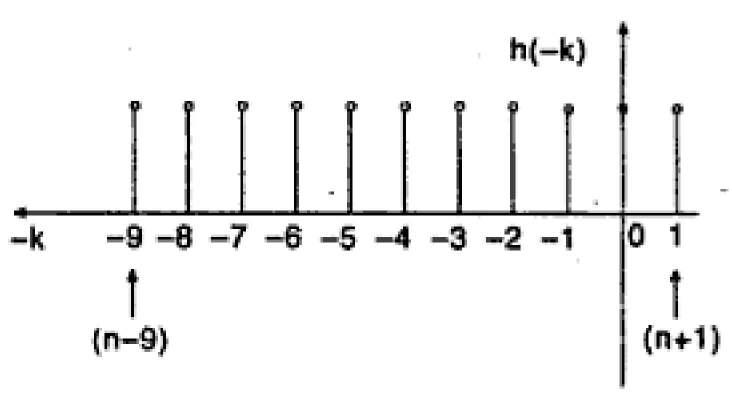

–u(n)
1 2 3 4 5 6 7 8 9
0
n
–1


2u(n–3)
2
1
0 1 2 3 4 5 6 7 8 9
n


–u(n–6)
1 2 3 4 5 6 7 8 9 10 11
0
n
–1


x(n) = –u(n)+2u(n–3)–u(n–6)
+1
1 2
0
3 4 5 6 7 8 9
n
–1


x(k)
+1
1 2
0
3 4 5 6 7
k
–1

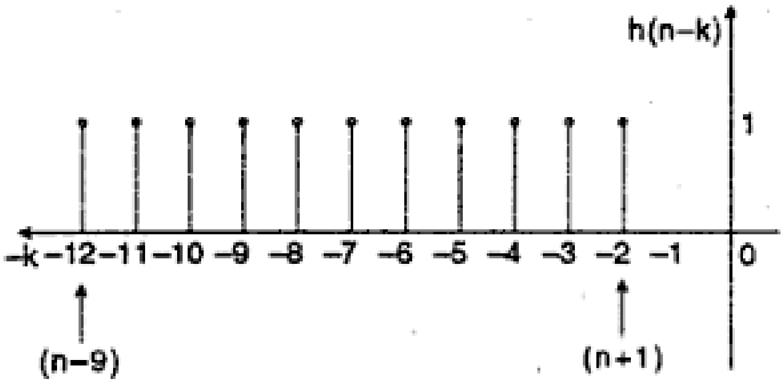

h(–k)
–k
–9 –8 –7 –6 –5 –4 –3 –2 –1
0 1
(n–9)
(n+1)

When $(n+1) < 0$

$$x(k) = \begin{cases} -1, & k = 0, 1, 2 \\ +1, & k = 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

$$h(n-k) = \begin{cases} 1, & k < 1 \\ 0, & \text{else where} \end{cases}$$

$$y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k)$$



Both $x(k)$ and $h(n-k)$ are not overlapped for $(n+1) < 0$. Therefore, the output response becomes zero.

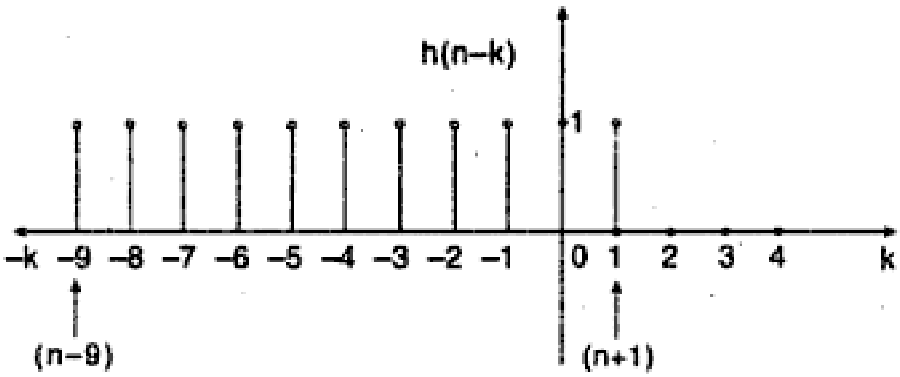
$$y(n) = 0$$

When $2 \geq (n+1) \geq 0 \Rightarrow 1 \geq n \geq -1$

$$y(n) = \sum_{k=0}^{(n+1)} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=0}^{n+1} (-1)(+1)$$

$$y(n) = -\sum_{k=0}^{n+1} 1 = -(n+2)$$



When $5 \geq (n+1) \geq 3 \Rightarrow 4 \geq n \geq 2$

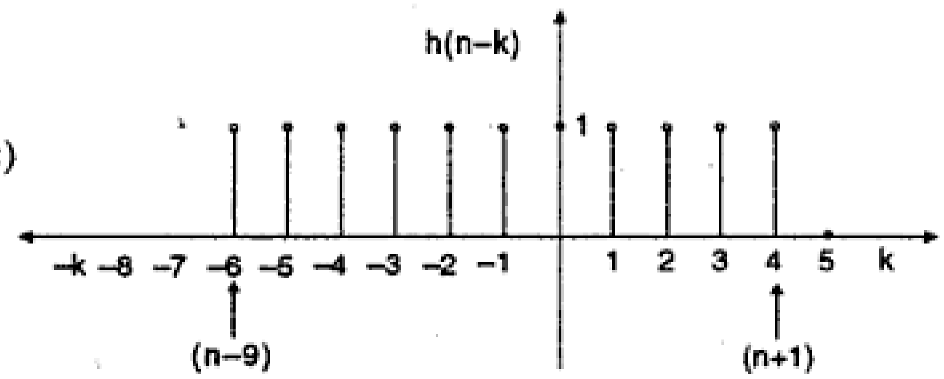
$$y(n) = \sum_{k=0}^{(n+1)} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=0}^{2} x(k)\,h(n-k) + \sum_{k=3}^{(n+1)} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=0}^{2} (-1)(+1) + \sum_{k=3}^{n+1} (1)(1)$$

$$y(n) = \sum_{k=0}^{2} (-1) + \sum_{k=3}^{n+1} (1)$$

$$y(n) = (n-4)$$

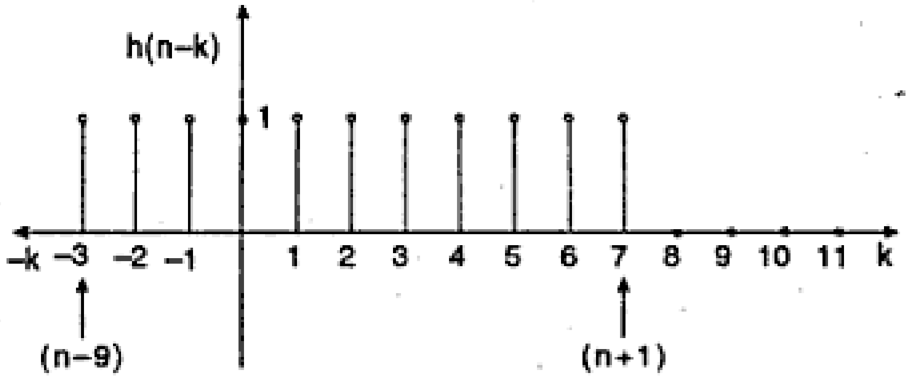


When $9 \geq (n+1) \geq 6 \Rightarrow 8 \geq n \geq 5$

$$y(n) = \sum_{k=0}^{5} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=0}^{2} x(k)\,h(n-k) + \sum_{k=3}^{5} x(n-k)\,h(n-k)$$

$$y(n) = \sum_{k=0}^{2} (-1) + \sum_{k=3}^{5} (1) = -3+3 = 0$$



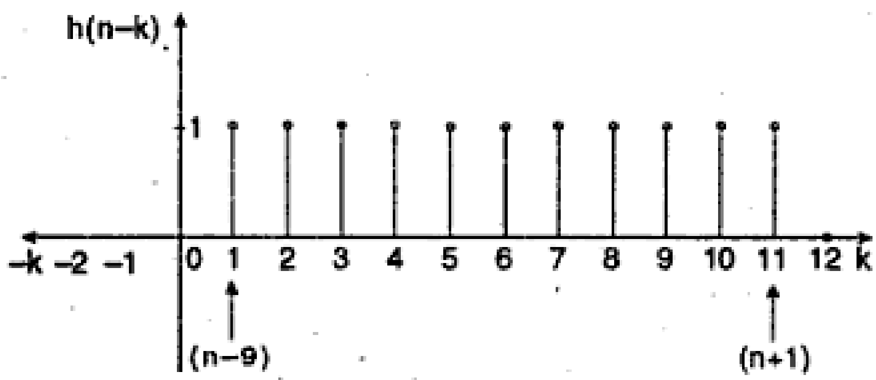
When $12 \geq (n+1) \geq 10 \Rightarrow 11 \geq n \geq 9$

$$y(n) = \sum_{k=(n-9)}^{5} x(k)\,h(n-k)$$

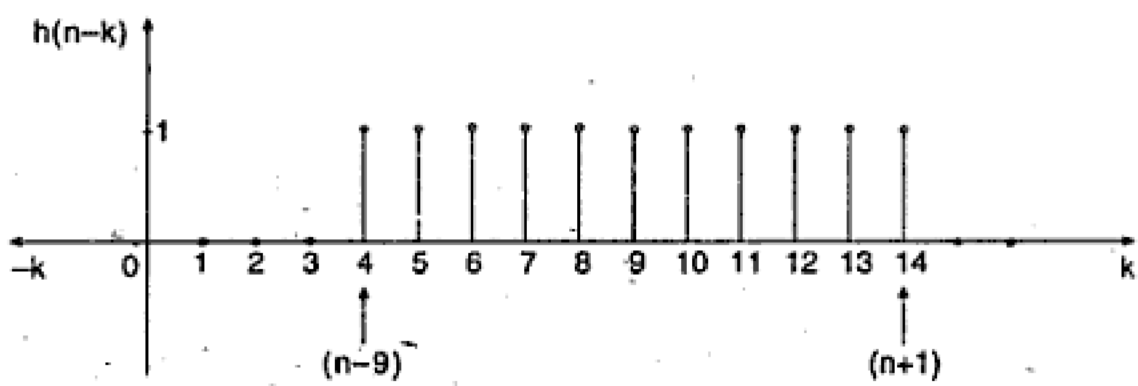
$$y(n) = \sum_{k=(n-9)}^{2} x(k)\,h(n-k) + \sum_{k=3}^{5} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=(n-9)}^{2} (-1) + \sum_{k=3}^{5} (1) = -[2+1-(n-9)]+3$$

$$y(n) = (n-9)$$



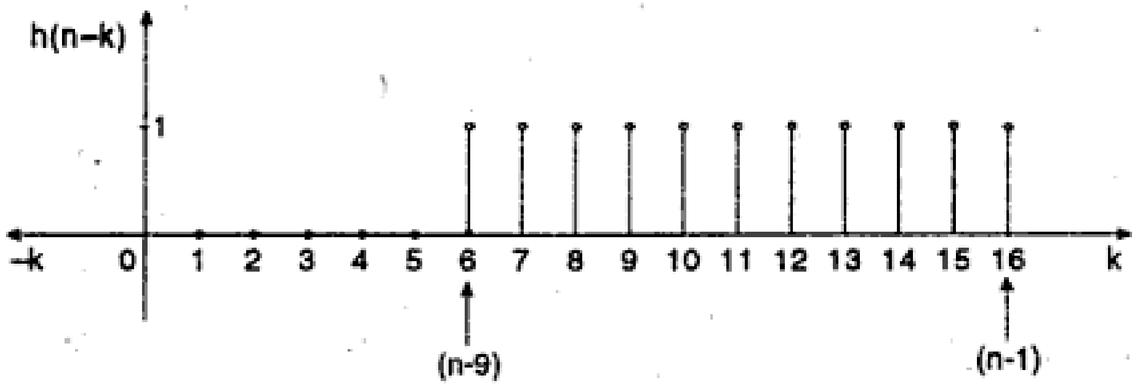
When $15 \geq (n+1) \geq 13 \Rightarrow 14 \geq n \geq 12$



$$y(n) = \sum_{k=(n-9)}^{5} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=(n-9)}^{5} (1) = -n+15$$

When $(n+1) \geq 16 \Rightarrow n \geq 15$



$$y(n) = 0$$

There is no overlapping of $x(k)$ with $h(n-k)$. Therefore, output response is zero.

$$y(n) = \begin{cases} 0 & n < -1 \\ -(n+2) & -1 \leq n \leq 1 \\ n+4 & 2 \leq n \leq 4 \\ 0 & 5 \leq n \leq 9 \\ n-9 & 10 \leq n \leq 11 \\ 15-9 & 12 \leq n \leq 14 \\ 0 & n > 14 \end{cases}$$

3.7 STEP RESPONSE

By using the convolution sum, we can easily represent the step response in terms of the impulse response. Let us consider the output response of the system $y(n)$ as,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} h(k) x(n-k) \tag{3.27}$$

where, $x(n)$ is input signal and $h(n)$ is impulse response of the system.

The step response of the system, means, applying a unit step function as a signal to the system, that is, $x(n) = u(n)$

where $$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad u(n-k) = \begin{cases} 1, & n \geq k \\ 0, & \text{otherwise} \end{cases}$$

Then, step response $$S(n) = \sum_{k=0}^{\infty} h(k) u(n-k)$$

$$S(n) = \sum_{k=0}^{n} h(k) \tag{3.28}$$

Equation (3.28) explains that the step response is the impulse response.

SOLVED PROBLEMS

Problem 3.19 Find the step response of the system if the impulse response is $h(n)=\alpha^n u(n), 0<\alpha<1$.

Solution The step response of the system is given by,

$$S(n)=h(n)*u(n)$$

$$S(n)=\sum_{k=-\infty}^{\infty} h(k)u(n-k)$$

$$S(n)=\sum_{k=-\infty}^{\infty} [\alpha^k u(k)]u(n-k)$$

$$S(n)=\sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Problem 3.20 Find the step response of the system if the impulse response $h(n)=\delta(n-2)-\delta(n-1)$.

Solution The step response, $\delta(n)=h(n)*u(n)$

$$S(n)=[\delta(n-2)-\delta(n-1)]*u(n)$$
$$S(n)=[\delta(n-2)*u(n)]-[\delta(n-1)*u(n)]$$
$$S(n)=u(n-2)-u(n-1)$$

■ 3.8 DECONVOLUTION

Deconvolution is "undo" procedure of convolution. In order to understand the deconvolution, let us consider an ideal system whose impulse response is $h(n)$ and system output is $y(n)$. This relation is expressed as,

$$y(n)=x(n)*h(n) \tag{3.29}$$

where $x(n)$ is the input signal.

The basic problem of deconvolution is to find $x(n)$ by deconvolute $h(n)$ with $y(n)$. Deconvolution has many practical applications, such as input pressure measurement by considering the output of the systems and system response.

By the definition of convolution sum as,

$$y(n)=\sum_{m=0}^{n} x(m)h(n-m) \tag{3.30}$$

Let us expand the convolution sum for $n=0,1,\ldots.\infty$

$$y(0)=x(0)\,h(0)$$
$$y(1)=x(0)\,h(1)+x(1)\,h(0)$$
$$y(2)=x(0)\,h(2)+x(1)\,h(1)+x(2)\,h(0)$$
$$y(3)=x(0)\,h(3)+x(1)\,h(2)+x(2)\,h(1)+x(3)\,h(0)$$
$$\vdots \qquad\qquad \vdots$$

The matrix form of above equations,

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & 0 & \cdots & 0 \\ h(1) & h(0) & 0 & 0 & & 0 \\ h(2) & h(1) & h(0) & 0 & & 0 \\ h(3) & h(2) & h(1) & h(0) & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \vdots \end{bmatrix} \qquad (3.31)$$

The input signal $x(n)$ can be directly computed by considering $h(n)$ and $y(n)$. The equations can be rewritten as

$$x(0) = \frac{y(0)}{h(0)}$$

$$x(1) = \frac{y(1) - x(0)h(1)}{h(0)}$$

$$x(2) = \frac{y(2) - x(0)h(2) - x(1)h(1)}{h(0)}$$

$$x(3) = \frac{y(3) - x(0)h(3) - x(1)h(2) - x(2)h(1)}{h(0)}$$

$$\vdots$$

$$x(n) = \frac{y(n) - \sum_{m=0}^{n-1} x(m)h(n-m)}{h(0)} \qquad (3.32)$$

Equation (4.32) is called the general deconvolution equation.

SOLVED PROBLEM

Problem 3.21 What is the input signal $x(n)$ that will generate the output sequence $y(n) = \{\underset{\uparrow}{1}, 5, 10, 11, 8, 4, 1\}$ for a system with impulse response $h(n) = \{\underset{\uparrow}{1}, 2, 1\}$ (A.U.April 2003)

Solution

$$y(n) = \{\underset{\uparrow}{1}, 5, 10, 11, 8, 4, 1\}\ ;\ h(n) = \{\underset{\uparrow}{1}, 2, 1\}$$

The total number of samples in the output response is $N_1 + N_2 - 1 = 7$.

The number of samples in the impulse response is $N_2 = 3$.

The number of samples in the input signal is $N_1 = 7 - N_2 + 1 = 5$.

Let us consider the general deconvolution equation

$$x(n) = \frac{y(n) - \sum_{m=0}^{n-1} x(m)h(n-m)}{h(0)}$$

For $n=0$, $$x(0) = \frac{y(0)}{h(0)} = \frac{1}{1} = 1$$

For $n=1$, $$x(1) = \frac{y(1) - x(0)h(1)}{h(0)} = \frac{5 - 1\times 2}{1} = 3$$

For $n=2$, $$x(2) = \frac{y(2) - x(0)h(2) - x(1)h(1)}{h(0)} = \frac{10 - 1\times 1 - 3\times 2}{1} = 3$$

For $n=3$, $$x(3) = \frac{y(3) - x(0)h(3) - x(1)h(2) - x(2)h(1)}{h(0)} = \frac{11 - 1\times 0 - 3\times 1 - 3\times 2}{1} = 2$$

For $n=4$, $$x(4) = \frac{y(4) - x(0)h(4) - x(1)h(3) - x(2)h(2) - x(3)h(1)}{h(0)}$$
$$= \frac{8 - 1\times 0 - 3\times 0 - 3\times 1 - 2\times 2}{1} = 1$$

The input sequence is $x(n) = \{1, 3, 3, 2, 1\}$.

3.9 BASIC SYSTEMS

The system can be broadly divided into two categories. They are

1. Finite impulse response system
2. Infinite impulse response system

Finite impulse response (FIR) system is one which exhibit zero response outside a finite time interval.

Example

$$h(n) = \begin{cases} 1, & |n| \le N \\ 0, & \text{otherwise} \end{cases}$$

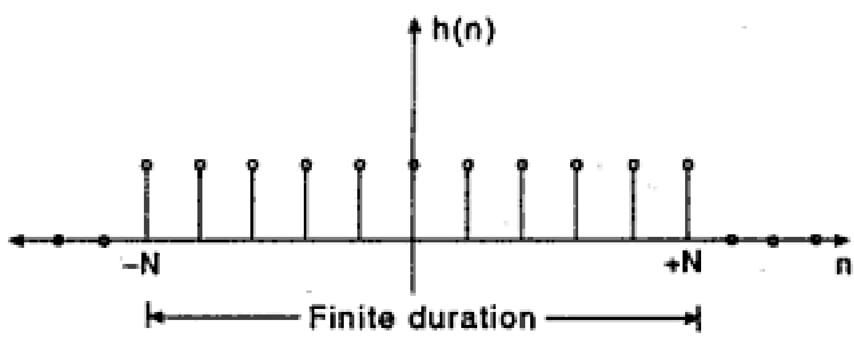


Fig. 3.10

Infinite impulse response (IIR) system is one which exhibit an impulse response of infinite duration.

Example

$$h(n) = \begin{cases} a^n, n \geq 0 \\ 0, \text{ otherwise} \end{cases}$$

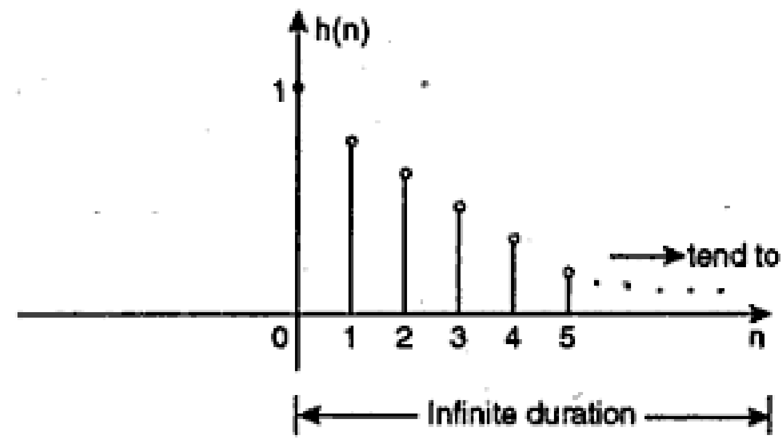


Fig. 3.11

3.10 LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATION

The difference equation generally represents the relationship between the input and output signals for a system. The Nth order difference equation is

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{3.33}$$

The equation (3.33) can be rewritten as

$$a_0 y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = \frac{1}{a_0} \sum_{k=0}^{M} b_k x(n-k) - \frac{1}{a_0} \sum_{k=1}^{N} a_k y(n-k) \tag{3.34}$$

where $a_0 = 1$, is a constant.

The output of the system can be obtained from the input and past output signal. This is clear in equation (3.34). Hence, for most discrete-time system implementations, we use equation (3.34).

3.10.1 Solution to Linear Constant Coefficient Difference Equation

It is convenient to express the output of the system described by a difference equation as a sum of two responses:

1. Natural Response ($y^{(n)}$)
2. Forced Response ($y^{(f)}$)

3.10.1.1 Natural response

The natural response of the system output can be obtained by considering the initial conditions. While calculating the natural response of the system output, the input is made zero. Therefore, the difference equation (3.33) is reduced to homogeneous equation, which is given by

$$\sum_{k=0}^{N} a_k y^{(n)}(n-k) = 0 \tag{3.35}$$

where superscript '(n)' represent the natural response. The natural response $y^{(n)}(n)$ is the solution to the homogeneous equation and is given by

$$y_h(n) = \lambda^n \tag{3.36}$$

where λ is N roots of the discrete-time system.

Substituting equation (3.36) in (3.35), we get

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \tag{3.37}$$

$$a_o\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_N\lambda^{n-N} = 0$$

$$\lambda^{n-N}(\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + a_N) = 0$$

Since $\lambda^{n-N} \neq 0$

$$\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + a_N = 0 \tag{3.38}$$

Equation (3.38) is the characteristic equation of Nth root.

Case (i) If roots $\lambda_1, \lambda_2, \lambda_3, \ldots$ are distinct.

The general solution to the homogeneous equation whose roots are distinct is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n + \ldots + C_N\lambda_N^n \tag{3.39}$$

Example If roots are distinct, say $\lambda_1 = 4$ and $\lambda_2 = 6$, then according to equation (3.39), the solution to the homogeneous equation is $y_h(n) = C_1(4)^n + C_2(6)^n$

Case (ii) If roots are repetitive.

The general solution to the homogeneous equation whose roots are repetitive m times is

$$y_h(n) = \lambda_1^n(C_1 + C_2n + C_3n^2 + C_4n^3 + \ldots + C_mn^{m-1}) \tag{3.40}$$

Example If $\lambda_1 = 8$ repeated 2 times and $\lambda_2 = 4$, then according to equation (3.40), the solution to homogeneous equation is

$$y_h(n) = (C_1 + C_2n)(8)^n + C_3(4)^n$$

Case (iii) If roots are complex.

The general solution to homogeneous equation whose roots are complex, i.e. $\lambda_1 = a + jb$ and $\lambda_2 = a - jb$ is

$$y_h(n) = r^n(A_1 \cos n\theta + A_2 \sin n\theta) \tag{3.41}$$

where $r = \sqrt{a^2 + b^2}$

$\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

A_1, A_2 = constant coefficients

Let $n=0$,

$$y(0)=\frac{5}{6}y(-1)-\frac{1}{6}y(-2)$$

$$y(0)=\frac{5}{6}-\frac{1}{6}=\frac{2}{3}$$

Let $n=1$,

$$y(1)=\frac{5}{6}y(0)-\frac{1}{6}y(-1)$$

$$y(1)=\frac{5}{6}\left(\frac{2}{3}\right)-\frac{1}{6}=\frac{7}{18}$$

Substitute the value of $y(0)$ and $y(1)$ in the equations (4) and (5) respectively, we get

$$C_1+C_2=\frac{2}{3}$$

$$\frac{1}{2}C_1+\frac{1}{3}C_2=\frac{7}{18}$$

On solving, we obtain $C_1=1;\ C_2=-1/3$

Substituting the values of C_1 and C_2 in equation (3)

$$y_h(n)=\left(\frac{1}{2}\right)^n-\frac{1}{3}\left(\frac{1}{3}\right)^n$$

which is the natural response of the given difference equation.

3.10.1.2 *Forced response*

The forced response of the system output can be obtained by considering the input signal alone (initial conditions are assumed to be zero). The forced response consists of two parts

(1) Homogeneous solution $[y_h(n)]$

(2) Particular solution $[y_p(n)]$

The particular solution is denoted by $y_p(n)$ and it represents the solutions to any difference equation when the input is given. The particular solution $y_p(n)$ can be obtained by considering the input signal $x(n)$, $n\geq 0$, i.e.

$$y_p(n)=-\sum_{k=1}^{N}a_k y_p(n-k)+\sum_{k=0}^{M}b_k x(n-k) \tag{3.42}$$

The following table shows particular solutions for some of the standard signals.

Table 3.1 Standard Particular Solution

Input Signal $x(t)$	**Particular Solution** $y_p(n)$
Step signal, A	K
$A\alpha^n$	$K\alpha^n$
An^α	$k_0 n^\alpha + k_1 n^{\alpha-1} + k_2 n^{\alpha-2} + \ldots + k_m$
$\left.\begin{matrix} A\cos(\omega n+\phi) \\ A\sin(\omega n+\phi) \end{matrix}\right\}$	$k_1 \cos \omega n + k_2 \sin \omega n$

The forced solution of the system is obtained by adding both the homogeneous and particular solutions.

SOLVED PROBLEMS

Problem 3.23 Find the forced response of the system described by the difference equation

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n) - x(n-1)$$

when the input signal is $x(n) = 2^n u(n)$.

Solution

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n) - x(n-1)$$

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) - x(n-1) \qquad (1)$$

For input signal $x(n) = 2^n u(n)$, the particular solution from the table 3.1 is of the form

$$y_p(n) = K2^n \qquad (2)$$

Substituting equation (2) in equation (1),

$$K2^n - \frac{5}{6}K2^{n-1} + \frac{1}{6}K2^{n-2} = 2^n - 2^{n-1} \qquad (3)$$

Obtain the value of 'K' of equation (3) by substituting any value of 'n'.

Let $n = 0$,

$$K2^0 - \frac{5}{6}K2^{-1} + \frac{1}{6}K2^{-2} = 2^0 - 2^{-1}$$

$$K = \frac{4}{5}$$

Therefore, the particular solution is given by

$$y_p(n) = \frac{4}{5}(2)^n \qquad (4)$$

The homogeneous solution (obtained from equation 3.36) is

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=0 \text{ [input signal, } x(n)=0]$$

The solution to the homogeneous equation is

$$y_h(n)=\lambda^n$$

Therefore,
$$\lambda^n-\frac{5}{6}\lambda^{n-1}+\frac{1}{6}\lambda^{n-2}=0$$

$$\lambda^{n-2}\left(\lambda^2-\frac{5}{6}\lambda+\frac{1}{6}\right)=0$$

Since $\lambda^{n-2}\neq 0$
$$\lambda^2-\frac{5}{6}\lambda+\frac{1}{6}=0$$

$$\left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{1}{3}\right)=0$$

$$\lambda_1=\frac{1}{2};\ \lambda_2=\frac{1}{3}$$

Since the roots are distinct, we use equation (3.39)

$$y_h(n)=C_1\left(\frac{1}{2}\right)^n+C_2\left(\frac{1}{3}\right)^n$$

The forced response is given by

$$y^{(f)}(n)=y_h(n)+y_p(n)$$

$$y^{(f)}(n)=\left[C_1\left(\frac{1}{2}\right)^n+C_2\left(\frac{1}{3}\right)^n\right]+\frac{4}{5}2^n \tag{5}$$

Let $n=0$,
$$y^{(f)}(0)=C_1+C_2+\frac{4}{5} \tag{6}$$

$n=1$,
$$y^{(f)}(1)=\frac{1}{2}C_1+\frac{1}{3}C_2+\frac{8}{5} \tag{7}$$

Let us consider equation (1),

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=x(n)-x(n-1)$$

For $n=0$,
$$y(0)-\frac{5}{6}y(-1)+\frac{1}{6}y(-2)=x(0)-x(-1) \tag{8}$$

Since initial conditions are assumed to be zero for calculating the forced response, $y(-1)=0$; $y(-2)=0$; $x(-1)=0$.

Consider the input signal, $x(n)=2^n$

For $n=0$, $x(0)=1$

Therefore, equation (8) can be written as

$$y(0)=1$$

Similarly, for $n = 1$ in equation (1),

$$y(1) - \frac{5}{6}y(0) + \frac{1}{6}y(-1) = x(1) - x(0)$$

$$y(1) - \frac{5}{6}(1) + 0 = 2 - 1 \qquad \text{[Since } x(1) = 2\text{]}$$

$$y(1) = \frac{11}{6}$$

Substituting the values of $y(0)$ and $y(1)$ in equations (6) and (7)

$$C_1 + C_2 + \frac{4}{5} = 1$$

$$\frac{C_1}{2} + \frac{C_2}{3} + \frac{8}{5} = \frac{11}{6}$$

$$C_1 = 1; C_2 = \frac{-4}{5}$$

Substituting the values of C_1 and C_2 in equation (5),

$$y^{(f)}(n) = \left(\frac{1}{2}\right)^n - \frac{4}{5}\left(\frac{1}{3}\right)^n + \frac{4}{5}2^n \quad n \geq 0$$

3.10.1.3 *Complete response*

The complete response of the system can be obtained by adding both natural response and forced response.

$$y(n) = y^{(n)}(n) + y^{(f)}(n) \tag{3.43}$$

The complete response can be obtained directly without separately finding the natural response and forced response as we did in Problems (3.18) and (3.19).

SOLVED PROBLEMS

Problem 3.24 Find the complete response of the system described by the difference equation

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n-1) - x(n-1)$$

when the input signal $x(n) = 2^n u(n)$. The initial conditions are $y(-1) = 1$ and $y(-2) = 1$.

Solution

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n) - x(n-1) \tag{1}$$

The complete response of the system

$$y(n) = y^{(n)}(n) + y^{(f)}(n)$$

To obtain natural response set input signal to zero, i.e.

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) = 0 \tag{2}$$

The solution to the homogeneous equation is of the form

$$y_h(n) = \lambda^n$$

Therefore, equation (2) becomes

$$\lambda^n - \frac{5}{6}\lambda^{n-1} + \frac{1}{6}\lambda^{n-2} = 0$$

$$\lambda^{n-2}\left(\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6}\right) = 0$$

$$\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0$$

$$\lambda_1 = \frac{1}{2}; \lambda_2 = \frac{1}{3}$$

The solution to the homogeneous equation is

$$y_h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{3}\right)^n \tag{3}$$

To obtain the forced response, consider the input signal $x(n)$, i.e.

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) - x(n-1) \tag{4}$$

The solution to the particular equation is of the form [from Table (3.1)]

$$y_p(n) = K2^n$$

Therefore, equation (4) becomes

$$K2^n - \frac{5}{6}K2^{n-1} + \frac{1}{6}K2^{n-2} = 2^n - 2^{n-1} \tag{5}$$

Obtain the value of 'K' of equation (5) by substituting any value of 'n', which results in

Let $n = 0$,

$$K2^0 - \frac{5}{6}K2^{-1} + \frac{1}{6}K2^{-2} = 2^0 - 2^{-1}$$

$$K = \frac{4}{5}$$

Therefore, the solution for the particular equation becomes

$$y_p(n) = \frac{4}{5}2^n \tag{6}$$

The complete response is sum of the homogeneous solution and particular solution,

$$y(n) = y_p(n) + y_h(n)$$

$$y(n) = \frac{4}{5}2^n + C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{3}\right)^n \tag{7}$$

For $n = 0$,
$$y(0) = \frac{4}{5} + C_1 + C_2 \tag{8}$$

For $n = 1$,
$$y(1) = \frac{8}{5} + C_1\left(\frac{1}{2}\right) + C_2\left(\frac{1}{3}\right) \tag{9}$$

To obtain the value of $y(0)$ and $y(1)$, let us reconsider the difference equation (1).

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n) - x(n-1)$$

$x(n) = 2^n$; $x(0) = 1$

$x(1) = 2$; $x(-1) = 0$ (not given)

For $n = 0$,
$$y(0) = \frac{5}{6}y(-1) - \frac{1}{6}y(-2) + x(0) - x(-1)$$

Now substituting initial conditions $y(-1) = 1$ and $y(-2) = 1$

$$y(0) = \frac{5}{6}(1) - \frac{1}{6}(1) + 1 - 0$$

$$y(0) = \frac{5}{3}$$

For $n = 1$,
$$y(1) = \frac{5}{6}y(0) - \frac{1}{6}y(-1) + x(1) - x(0)$$

Now substituting initial conditions $y(-1) = 1$ and $y(-2) = 1$

$$y(1) = \frac{5}{6}\left(\frac{5}{3}\right) - \frac{1}{6}(1) + 2 - 1$$

$$y(1) = \frac{20}{9}$$

Substituting the values of $y(0)$ and $y(1)$ in equations (8) and (9) respectively

$$\frac{5}{3} = \frac{4}{5} + C_1 + C_2$$

$$\frac{20}{9} = \frac{8}{5} + \frac{1}{2}C_1 + \frac{1}{3}C_2$$

On solving,
$$C_1 = 2; \; C_2 = -\frac{17}{15}$$

Therefore, the complete response of the system is given by

$$y(n) = \frac{4}{5}2^n + 2\left(\frac{1}{2}\right)^n - \frac{17}{15}\left(\frac{1}{3}\right)^n$$

Problem 3.25 Find the system response described by a difference equation

$$y(n)-\frac{1}{2}y(n-1)=x(n)$$

when the input signal $x(n)=\sin 2n$ Initial condition $y(-1)=1$.

Solution The difference equation is

$$y(n)-\frac{1}{2}y(n-1)=x(n)$$

The complete response of the system is

$$y(n)=y_h(n)+y_p(n)$$

To find the solution to the homogeneous equation, set the input signal $x(n)=0$.

$$y(n)-\frac{1}{2}y(n-1)=0$$

The solution to the homogeneous equation is

$$y_h(n)=\lambda^n$$

The homogeneous equation becomes

$$\lambda^n-\frac{1}{2}\lambda^{n-1}=0$$

$$\lambda^{n-1}\left(\lambda-\frac{1}{2}\right)=0$$

$$\lambda=\frac{1}{2}$$

Therefore, the solution of the homogeneous equation is

$$y_h(n)=C\left(\frac{1}{2}\right)^n$$

To obtain the forced solution let us consider the input signal, $x(n)=\sin 2n$

$$y(n)-\frac{1}{2}y(n-1)=x(n)$$

The solution to the particular equation is

$$y_p(n)=K_1\cos 2n+K_2\sin 2n$$

$$(K_1\cos 2n+K_2\sin 2n)-\frac{1}{2}[K_1\cos 2(n-1)+K_2\sin 2(n-1)]=\sin 2n$$

Hint $\cos(A-B)=\cos A\cos B+\sin A\sin B$
$\sin(A-B)=\sin A\cos B-\cos A\sin B$

$$K_1 \cos 2n + K_2 \sin 2n - \frac{1}{2}[K_1(\cos 2n \cos 2 + \sin 2n \sin 2) + K_2(\sin 2n \cos 2 - \cos 2n \sin 2)] = \sin 2n$$

$$K_1 \cos 2n + K_2 \sin 2n - \frac{K_1}{2}\cos 2n \cos 2 - \frac{K_1}{2}\sin 2n \sin 2 - \frac{K_2}{2}\sin 2n \cos 2 + \frac{K_2}{2}\cos 2n \sin 2 = \sin 2n$$

On comparing left hand side and right hand side terms,

$$K_1 \cos 2n - \frac{K_1}{2}\cos 2 \cos 2n + \frac{K_2}{2}\cos 2n \sin 2 = 0$$

$$\cos 2n\left(K_1 - \frac{K_1}{2}\cos 2 + \frac{K_2}{2}\sin 2\right) = 0$$

$$K_1 - \frac{K_1}{2}\cos 2 + \frac{K_2}{2}\sin 2 = 0$$

$$K_1\left(1 - \frac{\cos 2}{2}\right) + K_2\left(\frac{\sin 2}{2}\right) = 0$$

$$1.208K_1 + 0.4547K_2 = 0 \tag{1}$$

Similarly,

$$K_2 \sin 2n - \frac{K_1}{2}\sin 2 \sin 2n - \frac{K_2}{2}\cos 2 \sin 2n = \sin 2n$$

$$K_2 - \frac{K_1}{2}\sin 2 - \frac{K_2}{2}\cos 2 = 1$$

$$K_1\left(-\frac{\sin 2}{2}\right) + K_2\left(1 - \frac{\cos 2}{2}\right) = 1$$

$$-0.4547\,K_1 + 1.208\,K_2 = 1 \tag{2}$$

On solving equations (1) and (2),

$$K_1 = -0.2729\,;\, K_2 = 0.725$$

Therefore, the solution to the particular equation is

$$y_p(n) = -0.2729 \cos 2n + 0.725 \sin 2n \tag{3}$$

The complete response of the system is

$$y(n) = y_h(n) + y_p(n)$$

$$y(n) = C\left(\frac{1}{2}\right)^n - 0.2729 \cos 2n + 0.725 \sin 2n$$

For $n = 0$, $\quad y(0) = C - 0.2729 \quad (4)$

To find the value of $y(0)$, let us consider the difference equation

$$y(n)-\frac{1}{2}y(n-1)=x(n)$$

where $x(n)=\sin 2n$

For $n=0$, $x(0)=0$

$$y(0)=\frac{1}{2}y(-1)+x(0)$$

$$y(0)=\frac{1}{2} \tag{5}$$

Substitute the value of y(0) from equation (5) to (4)

$$\frac{1}{2}=C-0.2729 \quad \text{or} \quad C=0.7729$$

The complete response of the system is given by

$$y(n)=0.7729\left(\frac{1}{2}\right)^n-0.2729\cos 2n+0.725\sin 2n$$

Problem 3.26 Find the system response described by a difference equation

$$y(n)-\frac{7}{12}y(n-1)+\frac{1}{12}y(n-2)=2 \quad \text{for } n\geq 0$$

The initial conditions are $y(-1)=2$ and $y(-2)=3$.

Solution Given that

$$y(n)-\frac{7}{12}y(n-1)+\frac{1}{12}y(n-2)=2 \tag{1}$$

To obtain the natural response, set the input signal to zero.

The homogeneous equation becomes

$$y(n)-\frac{7}{12}y(n-1)+\frac{1}{12}y(n-2)=2 \tag{2}$$

The solution to the homogeneous equation is

$$y_h(n)=\lambda^n \tag{3}$$

The homogeneous equation becomes

$$\lambda^n-\frac{7}{12}\lambda^{n-1}+\frac{1}{12}\lambda^{n-2}=0$$

$$\lambda^{n-2}\left(\lambda^2-\frac{7}{12}\lambda+\frac{1}{12}\right)=0$$

Since $\lambda^{n-2} \neq 0$:
$$\lambda^2 - \frac{7}{12}\lambda + \frac{1}{12} = 0$$
$$\left(\lambda - \frac{1}{3}\right)\left(\lambda + \frac{1}{4}\right) = 0$$
$$\lambda_1 = \frac{1}{3};\ \lambda_2 = \frac{1}{4}$$

Since the roots are distinct, the solution to the homogeneous equation is

$$y_h(n) = C_1\left(\frac{1}{3}\right)^n + C_2\left(\frac{1}{4}\right)^n \quad (4)$$

To obtain the forced response, let us consider the input signal, which is a step signal whose amplitude is 2 , that is, $x(n)=2u(n)$.

The particular equation becomes

$$y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = 2u(n) \quad (5)$$

The solution to the particular solution [from Table (3.1)] becomes

$$y_p(n) = K\,u(n) \quad (6)$$

Substituting equation (6) in equation (5)

$$K\,u(n) - \frac{7}{12}K\,u(n-1) + \frac{1}{12}K\,u(n-2) = 2u(n)$$
$$K - \frac{7}{12}K + \frac{1}{12}K = 2$$
$$K = 4$$

Therefore, $\quad y_p(n) = 4\,u(n)$

The complete solution of the system becomes

$$y(n) = y_h(n) + y_p(n)$$
$$y(n) = C_1\left(\frac{1}{3}\right)^n + C_2\left(\frac{1}{4}\right)^n + 4\,u(n)$$

For $n=0$, $\quad y(0) = C_1 + C_2 + 4 \quad (7)$

For $n=1$, $\quad y(1) = \frac{1}{3}C_1 + \frac{1}{4}C_2 + 4 \quad (8)$

In order to find the values of $y(0)$ and $y(1)$, let us consider the difference equatio

$$y(n) = \frac{7}{12}y(n-1) - \frac{1}{12}y(n-2) + 2$$

For $n=0$, $\quad y(0) = \frac{7}{12}y(-1) - \frac{1}{12}y(-2) + 2$

$$y(0) = \frac{35}{12} = 2.9167$$

For $n = 1$,
$$y(1) = \frac{7}{12}y(0) - \frac{1}{12}y(-1) + 2$$
$$y(1) = \frac{7}{12}\left(\frac{35}{12}\right) - \frac{1}{12}(2) + 2 = 3.5347$$

Substitute the values of $y(0)$ and $y(1)$ in equations (7) and (8),
$$2.9167 = C + C + 4$$
$$3.5347 = \frac{1}{3}C + \frac{1}{4}C + 4$$

We obtain, $C_1 = 2.3337;\ C_2 = 1.2504$

Therefore, the complete response of the system is given by
$$y(n) = \left[2.3337\left(\frac{1}{3}\right)^n + 1.2504\left(\frac{1}{4}\right)^n + 4\right]u(n)$$

■ 3.11 INTRODUCTION TO CORRELATION

The correlation is another mathematical operation to measure the degree of similarity of any two signals/ images. Correlation is used in RADAR, digital communication, remote sensing engineering, etc.

Let us explain correlation with respect to the following example. The signal sequences $x(n)$ and $y(n)$ are the transmitted and received signals respectively.

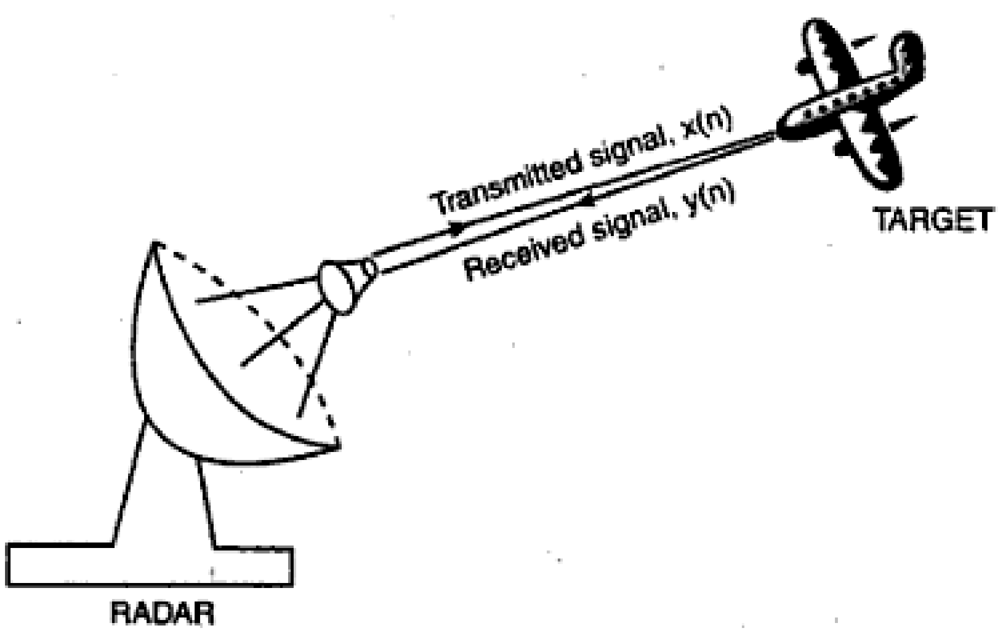


Fig. 3.12 Target Detection Using RADAR

If the target is present in the space during RADAR search, then the received signal $y(n)$ is the delayed input signal $x(n-D)$, i.e.,

$$y(n) = \alpha x(n-D) + w(n)$$

where α = attenuation of signal $x(n)$

$w(n)$ = additive noise pick up along with the signal at RADAR

D = delay factor

The delay is directly proportional to the distance between the RADAR and the target. In practice, the received signal $x(n-D)$ is heavily corrupted by the additive noise to the point where a visual inspection of $y(n)$ does not reveal the presence or absence of the desired signal. Correlation helps us to extract this important information from $y(n)$.

3.11.1 Cross-correlation

Let us consider two different signal sequences $x(n)$ and $y(n)$ which has finite energy. The cross-correlation of $x(n)$ and $y(n)$ is given by,

$$\gamma_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-m), \; m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{3.44}$$

or equivalently

$$\gamma_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n+m)\, y(n), \; m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{3.45}$$

where, m = lag parameter

The subscript parameter 'xy' in γ_{xy} indicate the direction in which the sequence is shifted by m. In equation (3.44), the signal $x(n)$ is unshifted while $y(n)$ is shifted by 'm' units to the right (m is positive). In equation (3.45) the signal $y(n)$ is unshifted while $x(n)$ is shifted by 'm' units to the left (m is negative). Both equations (3.44) and (3.45) are identical, that is, both relations yields identical cross-correlation sequence.

Reversing the roll of $x(n)$ and $y(n)$ in equations (3.44) and (3.45) result in equations (3.46) and (3.47) respectively.

$$\gamma_{yx}(m) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-m), \; m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{3.46}$$

or equivalently

$$\gamma_{yx}(m) = \sum_{n=-\infty}^{\infty} x(n) y(n+m), \; m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{3.47}$$

On comparing equation (3.44) with (3.45) or (3.46) with (3.47), we conclude that

$$\gamma_{xy} = \gamma_{yx}(-m) \tag{3.48}$$

It is clear from equation (3.48) that γ_{yx} is the folded version of γ_{xy}.

If the length of the sequence $x(n)$ is N_1 and the length of the sequence $y(n)$ is N_2 then total length of correlation sequence is $N_1 + N_2 - 1$.

The major computational difference between convolution and correlation is that in case of convolution, one of the sequence is folded, then shifted, then multiplied by the other sequence to form the product sequence for that shift and the product terms are added. In case of correlation, except folding all other process remain the same, that is, one of the sequence is shifted, then multiplied by the other sequence to form the product sequence for that shift and the product terms are added. Mathematically, the correlation and convolution can be related as

$$\gamma_{xy} = x(m) * y(-m) \tag{3.49}$$

SOLVED PROBLEM

Problem 3.27 Determine the cross-correlation sequence of the sequences

$$x(n) = \{1, \underset{\uparrow}{2}, 3, 4, 5\};\ y(n) = \{5, 6, 7, \underset{\uparrow}{8}, 9\}$$

Solution

Let us consider equation (3.44)

$$\gamma_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-m)$$

For the given problem equation (1) reduces to

$$\gamma_{xy}(m) = \sum_{n=-2}^{2} x(n)\, y(n-m)$$

When $m = 0$

$$\gamma_{xy}(0) = \sum_{n=-2}^{2} x(n)\, y(n)$$

For $m = 0$, the cross-correlation is the product of $x(n)$ and $y(n)$ and sum of all the products, that is,

$$\gamma_{xy}(0) = x(-2)\,y(-2) + x(-1)\,y(-1) + x(0)\,y(0) + x(1)\,y(1) + x(2)\,y(2)$$
$$\gamma_{xy}(0) = 1\times 5 + 2\times 6 + 3\times 7 + 4\times 8 + 5\times 9 = 115$$

When $m = 1$

$$\gamma_{xy}(1) = \sum_{n=-2}^{2} x(n)\, y(n-1)$$
$$\gamma_{xy}(1) = x(-2)\,y(-3) + x(-1)\,y(-2) + x(0)\,y(-1) + x(1)\,y(0) + x(2)\,y(1)$$
$$\gamma_{xy}(1) = 0 + 2\times 5 + 3\times 6 + 4\times 7 + 5\times 8 + 0 = 96$$

When $m = 2$

$$\gamma_{xy}(2) = \sum_{n=-2}^{2} x(n)\, y(n-2)$$
$$\gamma_{xy}(2) = x(-2)\,y(-4) + x(-1)\,y(-3) + x(0)\,y(-2) + x(1)\,y(-1) + x(2)\,y(0)$$
$$\gamma_{xy}(2) = 0 + 0 + 3\times 5 + 4\times 6 + 5\times 7 + 0 + 0 = 74$$

When $m = 3$

$$\gamma_{xy}(3) = \sum_{n=-2}^{2} x(n)\,y(n-3)$$

$$\gamma_{xy}(3) = x(-2)\,y(-5) + x(-1)\,y(-4) + x(0)\,y(-3) + x(1)\,y(-2) + x(2)\,y(-1)$$

$$\gamma_{xy}(3) = 0+0+0+4\times5+5\times6+0+0 = 50$$

When $m = 4$

$$\gamma_{xy}(4) = \sum_{n=-2}^{2} x(n)\,y(n-4)$$

$$\gamma_{xy}(4) = x(-2)\,y(-6) + x(-1)\,y(-5) + x(0)\,y(-4) + x(1)\,y(-3) + x(2)\,y(-2)$$

$$\gamma_{xy}(4) = 0+0+0+0+5\times5+0 = 25$$

When $m = 5$

$$\gamma_{xy}(5) = \sum_{n=-2}^{2} x(n)\,y(n-5)$$

$$\gamma_{xy}(5) = x(-2)\,y(-7) + x(-1)\,y(-6) + x(0)\,y(-5) + x(1)\,y(-4) + x(2)\,y(-3)$$

$$\gamma_{xy}(5) = 0$$

$$\gamma_{xy}(m \geq 5) = 0$$

When $m = -1$

$$\gamma_{xy}(-1) = \sum_{n=-2}^{2} x(n)\,y(n+1)$$

$$\gamma_{xy}(-1) = x(-2)\,y(-1) + x(-1)\,y(0) + x(0)\,y(1) + x(1)\,y(2) + x(2)\,y(3)$$

$$\gamma_{xy}(-1) = 1\times6+2\times7+3\times8+4\times9+0 = 80$$

When $m = -2$

$$\gamma_{xy}(-2) = \sum_{n=-2}^{2} x(n)\,y(n+2)$$

$$\gamma_{xy}(-2) = x(-2)\,y(0) + x(-1)\,y(1) + x(0)\,y(2) + x(1)\,y(3) + x(2)\,y(4)$$

$$\gamma_{xy}(-2) = 1\times7+2\times8+3\times9+0+0 = 50$$

When $m = -3$

$$\gamma_{xy}(-3) = \sum_{n=-2}^{2} x(n)\,y(n+3)$$

$$\gamma_{xy}(-3) = x(-2)\,y(1) + x(-1)\,y(2) + x(0)\,y(3) + x(1)\,y(4) + x(2)\,y(5)$$

$$\gamma_{xy}(-3) = 1\times8+2\times9+0+0+0 = 26$$

When $m = -4$

$$\gamma_{xy}(-4) = \sum_{n=-2}^{2} x(n)\,y(n+4)$$

$$\gamma_{xy}(-4) = x(-2)\,y(2) + x(-1)\,y(3) + x(0)\,y(4) + x(1)\,y(5) + x(2)\,y(6)$$

$$\gamma_{xy}(-4) = 1\times9+0+0+0+0 = 9$$

When $m = -5$

$$\gamma_{xy}(-5) = \sum_{n=-2}^{2} x(n)\, y(n+5)$$

$$\gamma_{xy}(-5) = x(-2)\, y(3) + x(-1)\, y(4) + x(0)\, y(5) + x(1)\, y(6) + x(2)\, y(7)$$

$$\gamma_{xy}(-5) = 0$$

$$\gamma_{xy}(m \le -5) = 0$$

Therefore, the cross-correlation sequence of $x(n)$ and $y(n)$ is

$$\gamma_{xy}(m) = \{9, 26, 50, 80, 115, 96, 74, 50, 25\}$$

3.11.2 Autocorrelation

If $y(n) = x(n)$ then equation (3.43) and (3.46) reduces to

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l) \tag{3.50}$$

or equivalently

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, x(n) \tag{3.51}$$

Equations (3.50) and (3.51) are the autocorrelation equations of the sequence.

3.11.3 Properties of Cross-correlation and Autocorrelation

Let us consider two data sequences $x(n)$ and $y(n)$ whose linear combination is given by

$$Z(n) = A\, x(n) + B\, y(n-l)$$

Where, A and B are scalar

l is the shift

The energy of the sequence $Z(n)$ is given by

$$E = \sum_{n=-\infty}^{\infty} Z^2(n)$$

$$E = \sum_{n=-\infty}^{\infty} [A\, x(n) + B\, y(n-l)]^2$$

On simplification,

$$E = A^2 \sum_{n=-\infty}^{\infty} x^2(n) + B^2 \sum_{n=-\infty}^{\infty} y^2(n-l) + 2AB \sum_{n=-\infty}^{\infty} x(n)\, y(n-l)$$

$$E = A^2 \gamma_{xx}(0) + B^2 \gamma_{yy}(0) + 2AB\gamma_{xy}(l)$$

If the energy of signals $x(n)$ and $y(n)$ are finite, then the energy of $Z(n)$ must also be finite, that is,

$$E = A^2 \gamma_{xx}(0) + B^2 \gamma_{yy}(0) + 2AB\gamma_{xy}(l) \ge 0$$

On simplification, by inequality

$$|\gamma_{xy}(l)| \le \sqrt{\gamma_{xx}(0)+\gamma_{yy}(0)} \quad \text{(Proof is left to the reader)}$$

If $x(n) = y(n)$, then

$$|\gamma_{xy}(l)| \le \sqrt{\gamma_{xx}(0)} = E_x$$

The autocorrelation of sequence attains maximum energy condition at zero lag (that is, $l = 0$).

SOLVED PROBLEM

Problem 3.28 Find the cross-correlation of two finite length sequences $x(n) = \{1,2,3,4\}$ and $y(n) = \{5,6,7,8\}$.

Solution

$x(l)=\{1,2,3,4\}$; $y(-l)=\{8,7,6,5\}$

By definition, $\gamma_{xy}(l)=x(l)*y(-l)$

x(l) \ y(–l)	8	7	6	5
1	8	7	6	5
2	16	14	12	10
3	24	21	18	15
4	32	28	24	20

$$\gamma_{xy}(l)=\{8, 16+7,\ 24+14+6,\ 32+21+12+5, 28+18+10,\ 24+15,\ 20\}$$
$$\gamma_{xy}(l)=\{8, 23, 44, 70, 56, 39, 20\}$$

CHAPTER SUMMARY

- Any signal can be represented as a time-shifted impulse sequence.
- The convolution gives the relation between input signal, impulse response and output response. It also explains how output response is obtained from input signal when it is passed through a system impulse response. The convolution sum is denoted by $y(n)=\sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n)*h(n)$.
- The LTI system satisfies basic three properties, that is, distributive, associative and commutative properties.
- Distributive property: $x(n)*[h_1(n)+h_2(n)] = x(n)*h_1(n)+x(n)*h_2(n)$

- Associative property: $[x(n)*h_1(n)]*h_2(n)=x(n)*[h_1(n)*h_2(n)]$
- Commutative property: $[x(n)*h(n)]=[h(n)*x(n)]$

- The procedure for linear convolution is as follows:
  1. Plot both $x(k)$ and $h(k)$.
  2. Reflect $h(k)$ about $k=0$ to obtain $h(-k)$.
  3. Shift $h(-k)$ by n(toward left).
  4. Let the initial value of n be negative.
  5. Multiply each element of $x(k)$ with $h(n-k)$ and add all the product terms to obtain $y(n)$.
  6. Shift $h(n-k)$ by incrementing the value of n by one and so step 5.
  7. Do step 6 until the product of $x(k)$ and $h(n-k)$ reduces to zero.
- The solution to the difference equation is analyzed by considering natural response and forced response.
- The natural response can be obtained by considering the initial values alone. The input signal to the system is assumed to be zero. The natural response can be obtained by solving homogeneous equation.
- The forced response can be obtained by considering the input signal alone. The initial values of the system are assumed to be zero. The forced response can be obtained by solving homogeneous equation and particular equation.
- The correlation is another mathematical operation to measure the degree to which the signals are similar. The autocorrelation refers to the correlation of same signal where as cross-correlation refers to the correlation of two different signals.
- The cross-correlation of two different signal sequences $x(n)$ and $y(n)$ which has finite energy is given by

$$\gamma_{xy}=\sum^{\infty} x(n)y(n-m),\quad m=0,\pm1,\pm2,\ldots$$

- The autocorrelation of a signal sequence $x(n)$ is given by $\gamma_{xx}(l)=\sum_{-\infty}^{\infty} x(n)x(n-l),\ \ l=0,\pm1,\pm2,\ldots$

REVIEW QUESTIONS

1. Define convolution sum.
2. Derive an expression for convolution sum.
3. Explain the properties of convolution sum.
4. Express the following signals in terms of impulse function.
   (a) $x(n)=\{0,2,4,8,10\}$
   (b) $x(n)=\{4,2,3,0,1,4,7,9,12\}$
5. Define invertibility of LTI system.
6. Define stability in LTI system.